CS4860 Lecture 25 Set Theory in FOL

Tue Nov. 27, 2012

Most of modern mathematics can be axiomatized over FOL. One of the most widely cited set of axioms for set theory is ZF, for Zermelo (1908) and Fraenkel (1922 replacement axiom scheme). ZFC is ZF with the axiom of choice (AC). The supplemental notes are Chapter I of an excellent book by Kenneth Kunen entitled Set Theory An Introduction to Independence Proofs. This chapter covers all the material covered in class.

The lecture stressed the ZF definition of the natural numbers $\mathbb{N}$. (Kunen does this on page 16, 18 and 19. On p.19 he states the Peano axioms as expressed in ZF.) Here are the natural numbers:

$\emptyset$ is the empty set (also denoted by $\{\}$). It is the number 0. $\{\emptyset\}$ is 1, $\{\emptyset, \{\emptyset\}\}$ is 2, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is 3. Notice that 1 is $\{0\}$, 2 is $\{0, 1\}$, 3 is $\{0, 1, 2\}$. In general n is $\{0, 1, \ldots, n-1\}$. This is very elegant. All of these sets are transitive, that is every element is also a subset. An ordinal number is a transitive set that is well-ordered by $\in$, the membership relation.

We noted in class that this definition is totally non-computational and non-constructive. There is a version of ZF axiomatized over iFOL, called IZF. It has some computational meaning, but for ZF over FOL there is "no notion of computation in sight." All computation is done in the meta-theory of ZF. That theory usually has computational ideas associated with proofs and induction on formulas and proofs. It can be formulated in iFOL.