20.1 A finite axiomatization of simple arithmetic.

Background

In 1971 Alfred Tarski, Andrzej Mostowski, and Raphael Robinson wrote the book *Undecidable Theories* which summarized in an elegant way the key undecidability and incompleteness results of Gödel, Church, and Tarski. Their *very simple theory, called* $\mathcal{Q}$, was featured in that book along with results of Julia Robinson showing that $\mathcal{Q}$ is undecidable. We might think of $\mathcal{Q}$ as Tarski/Mostowski/Robinson^2 (TMR^2 = $\mathcal{Q}$). In their textbook *Computability and Logic*,* George Boolos and Richard Jeffrey featured $\mathcal{Q}$. That book often uses highly non-constructive methods. We will stress computational approaches because the key theorems of Gödel and Church are completely constructive and deal with fundamental issues of constructive logics.

The theory $\mathcal{Q}$ is a finitely axiomatizable theory of addition and multiplication axiomatized in FOL with equality, $a = b$, as primitive and with a constant 0 and a successor operation used to define $s(0), s\big(s(0)\big), \ldots$. Thus there are *constants* for all the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. In the theory $\mathcal{Q}$ these are called *numerals* and we write $\overline{n}$ for a number n to denote its numeral in $\mathcal{Q}$, e.g. $\overline{1} = s(0), \overline{2} = s\big(s(0)\big), \ldots$ and we start with $\overline{0} = 0$.

*Cambridge University Press, Cambridge 1974, 1980, 1989 (3rd edition), 1991, 1992.

Axioms for $\mathcal{Q}$

To define $\mathcal{Q}$ in pure iFOL we add the atomic predicates $Eq(x, y)$, $Add(x, y, z)$ and $Mult(x, y, z)$. We write these informally as $x = y$, $x + y = z$, $x * y = z$. We take 0 as a constant and $s(t)$ as a one place function application. In principle we can avoid 0 and $s(\ )$ by introducing the additional predicates $Zero(x)$ and $Suc(x, y)$ for $y = s(x)$.

At the start we follow Boolos and Jeffrey. (They actually write t' for the successor of expression t which we write as $s(t)$ and they use $+, *$ as binary function symbols.)

		Axioms of $\mathcal{Q}$ (Boolos & Jeffrey)	FOL style
Define $s(\ )$	Ax 1.	$\forall x.\forall y.(x' = y' \supset x = y)$	$\forall x, y.\big(Eq\big(s(x), s(y)\big) \supset Eq(x, y)\big)$
	Ax 2.	$\forall x.\, 0 \neq x'$	$\forall x, y.\big(Suc(x, y) \supset \sim Zero(y)\big)$
	Ax 3.	$\forall x.(x \neq 0 \supset \exists y.\, x = y')$	$\forall x.\big(\sim Zero(x) \supset \exists y.\, Suc(y, x)\big)$
Define $+$	Ax 4.	$\forall x.(x + 0 = x)$	$\forall x, y.\big(Zero(y) \supset Add(x, y, x)\big)$
	Ax 5.	$\forall x, y.(x + y') = (x + y)'$	$\forall x, y, y', z, z'.\big(\big(Suc(y, y')\ \&\ Add(x, y', z)\ \&\ Add(x, y, z)\big) \supset Suc(z, z')\big)$
Define $*$	Ax 6.	$\forall x.(x * 0 = 0)$	$\forall x, y.\big(Zero(y) \supset Mult(x, y, y)\big)$
	Ax 7.	$\forall x, y.\big(x * y' = (x * y) + x\big)$	$\forall x, y, y', \mu, z.\big(\big(Suc(y, y')\ \&\ Mult(x, y, \mu)\ \&\ Mult(x, y', z)\big) \supset Add(\mu, x, z)\big)$

These axioms are very weak! We can't even prove very basic facts such as $\forall x.\big(x \neq s(x)\big)$, $\forall x.(0 + x = x)$, $\forall x.(0 * x = 0)$, $\forall x.(x < x')$, i.e. $\forall x.\big(x < s(x)\big)$ where $x < y$ iff $\exists z.(x + z = y \wedge z \neq 0)$.

But these axioms allow us to describe the behavior of *all recursive functions* in terms of their input/output behavior.

20.2 Representing recursive functions in $\mathcal{Q}$.

Definition: We say that a function f is *representable* in a FOL theory T by predicate $F(x_1, \ldots, x_n, y)$ if and only if:

(i) $f(n_1, \ldots, n_k) = m \Rightarrow \models_T F(\overline{n}_1, \ldots, \overline{n}_k, \overline{m})$ and

(ii) $\models_T \exists!\, y.\, F(\overline{n}_1, \ldots, \overline{n}_k, y)$

Boolos and Jeffrey say $f(n) = m$ implies $\models_T \forall y.\big(F(\overline{n}, y) \Leftrightarrow y = \overline{m}\big)$. (Note, it's stronger to prove $\forall x_1, \ldots, x_n.\, \exists! y.\, F(x_1, \ldots, x_n, y)$.)

We will sketch a constructive proof that all recursive functions are representable. *But first we compare this theory to HA.*

20.3 Heyting Arithmetic and Peano Arithmetic

The standard axiomatization of arithmetic in classical first-order logic is due to Peano in 1889, called *Peano Arithmetic* (PA). The constructive version in iFOL is called *Heyting Arithmetic* (HA). Here are the axioms similar to those given by Kleene in his 1952 book *Introduction to Metamathematics*, page 82.

Equality axioms

E1. $\forall x.(x = x)$
E2. $\forall x, y.(x = y \Leftrightarrow y = x)$
E3. $\forall x, y, z.\big(x = y \Rightarrow \big((y = z) \supset x = z\big)\big)$

Defining Successor

S1. $\forall x, y.\big(x = y \Leftrightarrow s(x) = s(y)\big)$
S2. $\forall x. \sim\big(s(x) = 0\big)$

Defining Addition

A1. $\forall x.(x + 0 = x)$
A2. $\forall x, y.\big(x + s(y) = s(x + y)\big)$

Defining Multiplication

M1. $\forall x.(x * 0 = 0)$
M2. $\forall x, y.\big(x * s(y) = (x * y) + x\big)$

Induction (an axiom schema)

$\Big(P(0) \,\&\, \forall x.\Big(P(x) \supset P\big(s(x)\big)\Big)\Big) \supset A(x)$

Exercise

Show that in HA we can prove $\forall x.(0 + x = x)$ and $\forall x.(0 * x = 0)$.