CS4860 Fall 2012 Lecture 7

Thur Sept. 13, 2012

Announcement. Students who did not receive at least 14 of the 24 points on HW1 should do the 5 (easy) supplemental problems on HW2 to add 5 more points.

Today's topic continues our investigation of the connection between the constructive logic IPC and the classical (or Boolean) logic PC, the logic presented by Smullyan.

Given a formula of PC such as $(P_1 \Rightarrow (P_1 \vee P_2))$ we note that it involves exactly two propositional variables (Smullyan p. 5, we use capital letters he uses lower case), P_1, P_2. For a general formula using $P_1, \dots, P_n$ as propositional variables we write $F(P_1, \dots, P_n)$ or for an even more compact notation, we write $F(\bar{P})$ where $\bar{P}$ is a sequence such as $P_1, \dots, P_n$.

Given $P_1, \dots, P_n$ as $\bar{P}$ we write Decide$(\bar{P})$ for the formula

$(P_1 \vee \neg P_1) \,\&\, (P_2 \vee \neg P_2) \,\&\, \dots \,\&\, (P_n \vee \neg P_n)$.

Definition Given formula $F(\bar{P})$ its Booleanized version is Decide$(\bar{P}) \Rightarrow F(\bar{P})$. For example, if $F(P_1, P_2) = (\neg(P_1 \wedge P_2)) \Rightarrow (\neg P_1 \vee \neg P_2)$ then Decide$(P_1, P_2) \Rightarrow \neg(P_1 \wedge P_2) \Rightarrow (\neg P_1 \vee \neg P_2)$ is its Booleanization. We sometimes say that $F(\bar{P})$ is grounded in Decide$(\bar{P})$.

Theorem: For any PC formula $F(\bar{P})$, Decide$(\bar{P}) \Rightarrow (F(\bar{P}) \vee \neg F(\bar{P}))$

We will carefully prove this later by induction on the depth of the formula $F(\bar{P})$ or the degree (Smullyan p. 8). You have proved the base case already in HW1.

<u>Definition</u> Given a proof of $Decide(\bar{P}) \Rightarrow F(\bar{P})$ or an evidence term evd for this IPC formula (same as a PC formula), we say that it is of dimension $d \leq n$ where $\bar{P} = P_1, \ldots, P_n$ iff it uses exactly d of the $(P_i \vee \neg P_i)$ instances.

<u>Theorem</u> If the evidence term for $Decide(\bar{P}) \Rightarrow F(\bar{P})$ is of dimension 0, then $F(\bar{P})$ is constructively valid. If there is no evidence term d of dimension 0 but there is one of dimension $d > 0$, then $F(\bar{P})$ is classically valid but not constructively valid. A simple example is $F(P_1) = P_1 \vee \neg P_1$, another is $(P_1 \Rightarrow P_2) \Rightarrow \neg P_1 \vee P_2$.

We will now take a look at how truth tables can provide evidence terms for any Booleanized IPC formula. This gives us a simple way to relate Boolean (classical) truth to constructive truth. It is also a way to build evidence terms for all classically true formulas that are Booleanized. This method of truth-table evidence (TT-evidence) is a way to build evidence terms systematically from truth tables.

Truth Table Evidence

We discussed this topic already in Lecture 6, and Prof. Kreitz wrote excellent notes. These notes mainly give one example in great detail using the new terminology introduced in this lecture.

Consider the truth table for $P \vee Q$ (we use P, Q instead of P_1, P_2 for easier readability, think of $P = P_1$, $Q = P_2$ in the above.)

P	Q	$P \vee Q$	$P \wedge Q$	$P \supset Q$	$P \supset False$	
t	t	t	t	t	f	
t	f	t	f	f	f	
f	t	t	f	t	t	
f	f	f	f	t	t	

Now consider how the truth table relates to evidence in one case, the $P \vee Q$ case.

	$P \vee \neg P$ P		$Q \vee \neg Q$ Q	$(P \vee Q) \vee \neg(P \vee Q)$ $P \vee Q$
t	$inl(p)$	t	$inl(q)$	$inl(inl(p))$ $inl(inl(q))$
t	$inl(p)$	f	$inr(nq)$	$inl(inl(p))$
f	$inr(np)$	t	$inl(q)$	$inl(inl(q))$
f	$inr(np)$	f	$inr(nq)$	$inr(\lambda x.\ case(x)\ is\ \begin{array}{l} inl(p) \to np(p) \\ inr(q) \to nq(q) \end{array})$

where $np = \lambda x.x$

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We consider now a truth-table analysis of a problem you solved in HW1, showing $(P \vee \neg P) \,\&\, (Q \vee \neg Q) \Rightarrow ((P \vee Q) \vee \neg(P \vee Q))$.

P	Q	$(P \vee Q) \vee \neg(P \vee Q)$
$inl(p)$	$inl(q)$	$inl(inl(p))$ $inl(inl(q))$
$inl(p)$	$inr(nq)$	$inl(inl(p))$
$inr(np)$	$inl(q)$	$inl(inl(q))$
$inr(np)$	$inr(nq)$	$inr(\lambda x.\ case(x)\ is\ inl(p) \to np(p) \mid inr(q) \to nq(q))$

note in $\lambda x. -$, $x : (P \vee Q)$

Now using the truth-table evidence we can constructively prove $(P \vee \neg P) \,\&\, (Q \vee \neg Q) \Rightarrow ((P \vee Q) \vee \neg(P \vee Q))$. We will use the truth-table to simply read off the evidence term (and calculate its dimension).

$$\lambda a.\ case(a_1)\ is \begin{cases} inl(p) \to case(a_2)\ is \begin{cases} inl(q) \to \underline{inl(inl(p))} \\ inr(nq) \to inl(inl(p)) \end{cases} \\ inr(np) \to case(a_2)\ is \begin{cases} inl(q) \to inl(inl(q)) \\ inr(nq) \to inr(\quad) \end{cases} \end{cases}$$

note $a : (P \vee \neg P) \,\&\, (Q \vee \neg Q)$
$a_1 : (P \vee \neg P)$
$a_2 : (Q \vee \neg Q)$

see term from the last row of the truth table above

$\lambda x.\ case(x)\ is\ inl(p) \to np(p)$
$\quad inr(q) \to nq(q)$

This evidence has dimension 2.

Here is a simpler example.

P	$(\neg\neg P \Rightarrow P) \vee \neg(\neg\neg P \Rightarrow P)$
$inl(p)$	$inl(\lambda x. p)$
$inr(np)$	$inl(\lambda f. any(f(np)))$

Note $\lambda x.\ case(x)\ is\ inl(p) \to inl(\lambda x. p) \mid inr(np) \to inl(\lambda f. any(f(np)))$ is the evidence for $(P \vee \neg P) \Rightarrow (\neg\neg P \Rightarrow P)$.

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Applying the TT-evidence

Consider the formula $(\neg Q \Rightarrow \neg P) \Rightarrow (P \Rightarrow Q)$. This is classically valid (a tautology – see Smullyan p. 11, Lecture 6 supplemental notes). Write the truth table to see the validity. We can also see the evidence easily from the Booleanized formula

$$(P \vee \neg P) \& (Q \vee \neg Q) \Rightarrow ((\neg Q \Rightarrow \neg P) \Rightarrow (P \Rightarrow Q))$$

The evidence term will be this for $a: (P \vee \neg P) \& (Q \vee \neg Q)$, $f: [\neg Q \Rightarrow \neg P)$

$$\lambda a. \lambda f. (\text{case}(a_2) \text{ is } inl(q) \rightarrow \lambda x. q \mid inr(nq) \rightarrow \lambda x. any(f(nq)(p)))$$

We see that this evidence term is 1-dimensional. We can actually prove $(Q \vee \neg Q) \Rightarrow ((\neg Q \Rightarrow \neg P) \Rightarrow (P \Rightarrow Q))$

P	Q	$\neg Q \Rightarrow \neg P$	$(\neg Q \Rightarrow \neg P) \Rightarrow (P \Rightarrow Q)$	$(P \Rightarrow Q)$
$inl(p)$	$inl(q)$	$f: [\neg Q \Rightarrow \neg P]$		$\lambda x. q$
$inl(p)$	$inr(nq)$	" "		$\lambda x. any(f(nq)(x))$
$inr(np)$	$inl(q)$			
$inr(np)$	$inr(nq)$			

Note this f is not a truth value but the function in $\neg Q \Rightarrow \neg P$

Note $\neg Q \Rightarrow \neg P$ is: $(Q \Rightarrow False) \Rightarrow (P \Rightarrow False)$

In $\lambda x. any(f(nq)(x))$ we know that $f(nq) \in [P \Rightarrow False]$, so for $x \in [P]$, $f(nq)(x) \in False$, thus $any(f(nq)(x)) \in [Q]$.