Applied Logic	**Homework 5**
CS 4860 Fall 2012	**Due Date: Nov 1, 2012**

Questions

(1) Please write up answers to any problems you missed on the Prelim. Solutions were given in class on Thur Oct 18. If you participated in that class you will get credit that way, as part of the 10% class participation part of the course grade, and you need not write up this problem.

(2) Solve Exercise 3 on page 50 of Smullyan.

(3) Provide evidence terms for these formulas. Assume C is a closed formula (thus no free occurrence of x).

- $\exists x.(C \supset Px) \supset (C \supset \exists x.Px)$
- $\forall x.(C \supset Px) \supset (C \supset \forall x.Px)$
- $(\exists y.\forall x.A(x,y) \supset \exists z.\forall x.B(x,z)) \supset \forall y.\exists z.(\forall x.A(x,y) \supset \forall x.B(x,z))$

(4) Note that we claim that the Fan Theorem is constructive. Give a method to compute the bound on the depth of the tree given a barring relation R.

(5) (a) Show that Hintikka's Lemma on page 58 is constructively true.

(b) Discuss why the Completeness Theorem on page 60 is not constructively true.

(6) On uniform validity

Recall our proof that $P \vee \sim P$ is not uniformly valid and hence not provable by Refinement Logic. We can use the special *uniform quantifier* $\forall[P].(P \vee \sim P)$ to state that there is evidence evd which belongs to $(P \vee \sim P)$ independently of what P is.

Use this technique to show $\sim \forall[A,B].((A \supset B) \supset A) \supset A$ similarly to how we proved $\sim \forall[P].(P \vee \sim P)$.

Hint: Show that $\forall[P].(P \vee \sim P)$ would follow from this. Substitute $(P \vee \sim P)$ for A and $False$ for B and recall the fact that we can prove $\sim\sim (P \vee \sim P)$. This will allow us to conclude $P \vee \sim P$ which we know is not uniformly true.

(7) Expressing properties of numbers in FOL.

Soon we will be studying properties of the natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ using logic. The theory of the integers, $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ has nicer algebraic properties than $\mathbb{N}$, and it is easy to express them in FOL, as this exercise will demonstrate.

The algebraic structure is based on properties of $+$ and 0 and multiplication, $*$, and 1. Each operator $+$ with 0 and $*$ with 1 forms a structure called a *commutative semi-group* satisfying these axioms where *op* is the operator ($+$ or $*$) and u is the unit (0 or 1).

Ax 1.	Associativity	$(x\ op\ y)\ op\ z = x\ op\ (y\ op\ z)$
Ax 2.	Commutativity	$(x\ op\ y) = (y\ op\ x)$
Ax 3.	Unit	$(x\ op\ u) = x = (u\ op\ x)$

Addition satisfies the property of having inverse elements, $-x$, satisfying

Ax 4.	Inverse	$x + (-x) = 0.$

This makes $\langle \mathbb{Z}, +, 0, - \rangle$ into a *commutative group*.

To express these facts in FOL we introduce an equality relation $Eq(x, y)$ satisfying

Reflexivity:	$Eq(x, x)$
Symmetry:	$Eq(x, y) \Rightarrow Eq(y, x)$
Transitivity:	$Eq(x, y)\ \&\ Eq(y, z) \Rightarrow Eq(x, z)$.

Express the laws defined above in FOL using $Add(x, y, z)$ for $x + y = z$ and $Mult(x, y, z)$ for $x * y = z$. Use predicates $Zero(x)$, $One(x)$ to define 0 and 1 and the relation $Inv(x, y)$ to say that x is the inverse of y (e.g. $x = -y$).

(a) Express the commutative semi-group axioms in FOL and the existence of additive inverses.

(b) Prove $\forall x, y, z.((Add(x, y, z)\ \&\ Eq(z, x)) \Rightarrow Zero(y))$ in iFOL.
Note, we can assume that all these predicates are *decidable*, e.g.
$\forall x.(Zero(x) \vee \sim Zero(x))$, $\forall x, y.(Eq(x, y) \vee \sim Eq(x, y))$,
$\forall x, y, z.(Add(x, y, z) \vee \sim Add(x, y, z))$.