Problem 1

(a) Give types for the following objects using B for Booleans, Form for Formulas, Var for Variables, Ops for the type {not, or, and, imp} of logical operators, N for natural numbers.

(i) functions from formulas to their outer operators
(ii) functions from formulas to Booleans
(iii) ordered pairs of Formulas and their depth
(iv) valuations of the type of all formulas (see pages 9 and 10)
(v) interpretations of a formula X (see page 10)

(b) Using these types write a definition of tautologies as a subtype of Form, use the set type notation from lecture, \{x:T | P(x)\}.

Problem 2

(a) Solve Exercise 3 page 13 of textbook

(b) Solve Exercise 3 for Conjunctive Normal Form (hint use part (a)).

Problem 3

Solve Exercise 5 page 14.

Problem 4

Construct tableau proofs or falsifications for these formulas (→ is implication)

(i) \(~( (p \text{ and } q) \text{ or } (\neg p \text{ and } r)) \rightarrow ((\neg p \text{ or } \neg q) \text{ and } (p \text{ or } \neg r))
(ii) \((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))
(iii) \((p \rightarrow (q \text{ or } r)) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r ))

Extra Credit Problems

Problem 5

Write a more detailed definition of Smullyan’s trees following the outline from class and prove the theorem that for every node x in S, there is a unique path p(x) such that end(p(x)) = x. Try to do the proof so that by following it, a reader can actually construct the path. Note in your proof where you use the fact that you can decided whether two points of S are equal. This proof will require you to know that every interior point (a point not equal to the origin) has a unique predecessor.

Problem 6

Write a recursive data type or define a recursive set that corresponds a close as possible to Smullyan’s simple definition of formulas on page 7. Based on this data type or set, define a recursive function which is a valuation for any formula given an interpretation for its variables.
Extra Reading: Look at the paper Expressing and Implementing the Computational Content Implicit in Smullyan’s Account of Boolean Valuations. It is posted as part of Lecture 3.