Reading: Please read Smullyan Chapter II pages 15–24.

1 Review

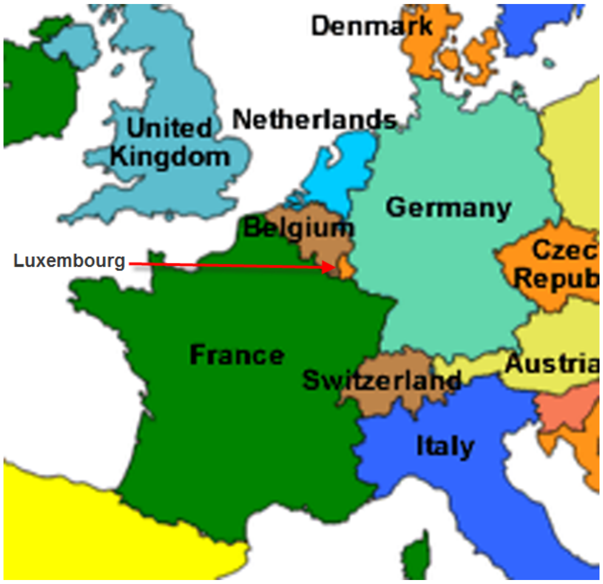
Here is our current ”location” on the course map from Lecture 1

Propositional Logic
Syntax(formulas) lecture 2/book Chapter $1 \S 1$.
Boolean valuations, interpretations, truth tables.
Value of a formula under an interpretation lecture 2,3 / book Chapter $1 \S 2$.
Algebra CNF, DNF, distributive laws lecture 2.
Encoding graph problems into CNF lecture 2.
SAT Problem lectures 2,3.
DPLL class of algorithms lecture 3,4.
SAT is NP this lecture.
Random SAT examples, phase transition, spinglass, Bart Selman’s results, this lecture.
SAT and AI - encoding graph coloring this lecture.

proofs (Chapter II $\S$ 1)
Justifying true propositions
proofs from axioms, modus ponens, substitution proof as a record of failed
falsification (Tableaux) rules schemes especially $X \supset Y$ rules
judgements and signed formulas $\vdash X$, FX

2 Encoding Graph Coloring

Consider the following encoding of graph coloring into CNF formulas.
The task is to color planar graphs so that no adjacent countries have the same color. Here is an example from the map of Europe.



Let the countries with the color i be B_i, G_i, F_i, L_i. Consider only three colors, red(r), blue(b), green(g). Then the possible coloring of France is $F_r \vee F_b \vee F_g$.

The constraints on pairs of neighbors derived from the graph include pairs such as $\sim(B_b \wedge L_b), \ \sim(B_r \wedge L_r)$, etc. These are equivalent to $(\sim B_b \vee \sim L_b) \wedge (\sim B_r \vee \sim L_r) \wedge$ etc.

Write down the CNF claim that red, green, and blue suffice to color this map and prove it is unsatisfiable.

3 SAT as an NP Problem

SAT is the first of hundreds of combinatorial problems shown to be *solvable in Nondeterministic Polynomial time (NP)*. Consider a problem to be a set S of elements from a discrete set U. S belongs to the class of NP problems if and only if there is a polynomial time algorithm R on U and on another discrete set T and a polynomial p such that

$$x \in S \ \ iff \ \ \exists\, t : T.|t| \leq p\,(|x|)\&R(x,t).$$

The running time of $R(x,t)$ is bounded by $c \cdot (|t| + |x|)^d$ where $|x|$ and $|t|$ is the length of x and t (think of x and t as strings of symbols) and c and are positive integers. We call t a *certificate* for x.

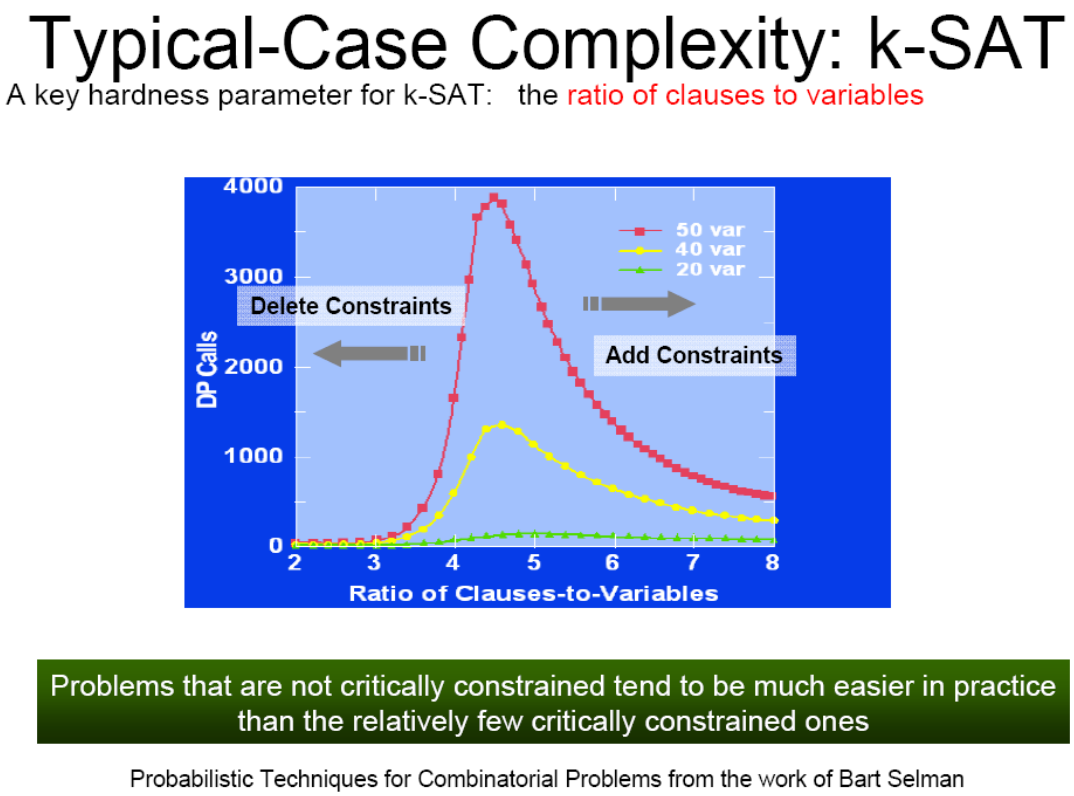
To see that SAT belongs to NP, let U be the set of propositional formulas, Form. Note, $SAT \subseteq Form$.

$$F \in SAT \; iff \; \exists a : Assignment(Var(F)).val(F, a) = true$$

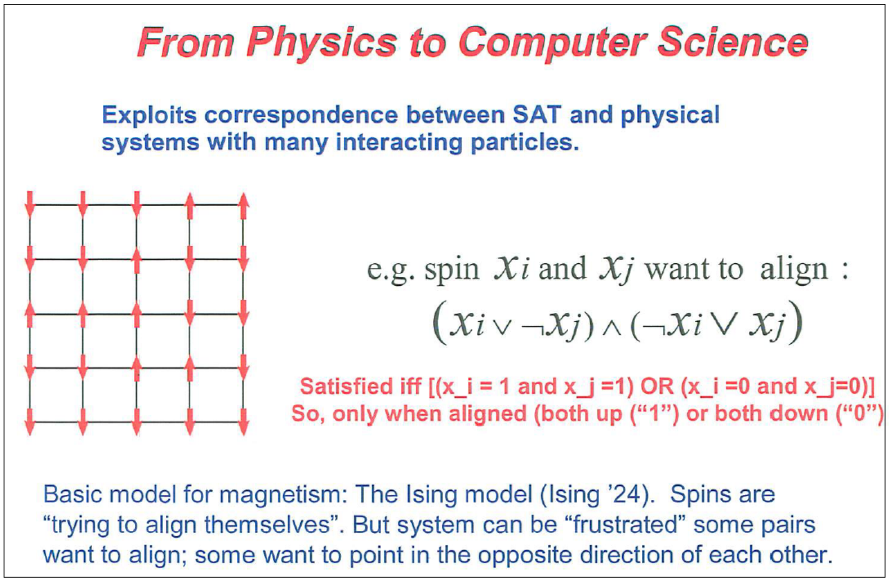
The assignment (or interpretation) a is bounded by twice the number of variables in F, and the Boolean validation val runs in time proportional to the depth of the formula F. The assignment a is the *certificate* for F.

4 Random SAT Problems

4.1 Typical-Case Complexity: k-SAT



4.2 SAT and Physics



5 Proofs

5.1 Church's Propositional Calculus P_2

Here is Church's well known "Hilbert-style" axiom system for propositional logic. It is in the axiomatic style of Euclid as made rigorous by Hilbert.

Rules of Inference

Modus Ponens   Substitution x a propositional variable

$$\frac{A,\ A \supset B}{B} \qquad \frac{A}{A\,[B\,/\,x]}$$

Axioms

(1) $p \supset (q \supset p)$

(2) $(s \supset (p \supset q)) \supset ((s \supset p) \supset (s \supset q))$

(3) $(\sim p \supset \sim q) \supset (q \supset p)$

A proof is a finite sequence of formulas $F_1, F_2, ..., F_n$ such that each F_i is an axiom or follows from previous formulas by a rule of inference.

5.2 Tableau Proofs

5.3 Example of a Tableau Proof

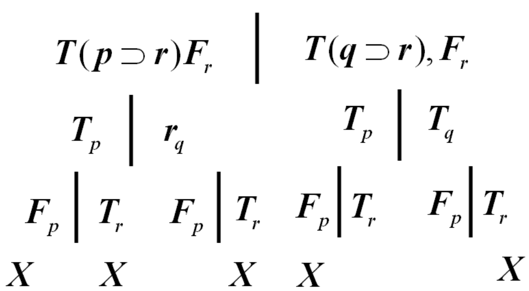
Either price-theory implies a recession or quant-theory implies one, if the two theories is true, there will be a recession.

$((p \supset r) \vee (q \supset r)) \supset ((p \vee q) \supset r)$

Here is a proof attempt using tableau rules

$F(((p \supset r) \vee (q \supset r)) \supset ((p \vee q) \supset r))$
$T((p \supset r) \vee (q \supset r)), F((p \vee q) \supset r))$
$T(p \vee q),\ Fr$

$F(X \supset Y)$ case
$TX,\ FY$
$F(X \supset Y)$ case again
$T(X \vee Y)$ case
$TX \mid TY$



Consider the assignment F_p, T_q, F_r

$$
\begin{gathered}
((f \supset f) \vee (t \supset f)) \supset ((f \vee t) \supset f) \\
(t \vee f) \supset (t \supset f) \\
t \supset f \\
f
\end{gathered}
$$

So price theory is false, quant theory is true, there is no recession so price-theory $\supset$ recession is vacuously true, but price theory or quant theory implies a recession is false, because quant theory implies a recession is false, even though quant theory is true.