Problem Set 9

Due Date: Thurs, April 17, 2003

Problems

1. a) Give a finite model of a semigroup that is not commutative.
   b) Give a finite model of a commutative semigroup that is not a monoid.

2. Consider the boolean ring \( \langle \mathbb{B}, =, \rightarrow, \lor, T, F \rangle \). Define the operations \( \sim, \land, \text{ and } \supset \) in terms of the ring operations and prove the following laws solely on the basis of the ring axioms.
   (1) \( p \supset (p \lor q) \),
   (2) \( (p \land q) \supset p \),
   (3) \( (p \land q) \supset q \),
   (4) \( p \supset (q \supset p) \),
   (5) \( \sim q \supset (q \supset p) \),
   (6) \( p \supset q \supset (\sim q \supset \sim p) \), and
   (7) \( p \lor p \equiv p \).

3. Is \( \langle \mathbb{Z}, =5, +, \ast \rangle \) a field?
   If so, give brief proofs of the axioms. If not, show which axiom is not satisfied.

4. Define \( x < y \equiv (\exists z)(x+z+1 = y) \) and prove the seven axioms of discrete linear orders for \(< \) from the Peano axioms.
   \( \text{lt-asym: } (\forall x,y) (x < y \supset \sim (y < x)) \)
   \( \text{lt-trans: } (\forall x,y,z) ((x < y \land y < z) \supset x < z) \)
   \( \text{lt-linear: } (\forall x,y) (x < y \lor y < x \lor x = y) \)
   \( \text{lt-discrete: } (\forall x,y) \sim (x < y \land y < x+1) \)
   \( \text{lt-0-1: } 0 < 1 \)
   \( \text{lt-mono++: } (\forall x,y,z) (x < y \supset x+z < y+z) \)
   \( \text{lt-mono--: } (\forall x,y,z) ((0 < z \land x < y) \supset x*z < y*z) \)