Assignment 5

Problem 2 (ii)

$$\forall p.(p \supset q) \supset \forall r.r$$

If you will try to prove this formula some proofs will fail. But one failed prove does not imply that the formula is invalid. Probably there is another proof of this formula. Actually if you would prove that *all* proofs would fail, then since P^2 is complete, you can conclude that this formula is invalid. (Note that this argument works only for *complete* logics). But the better way to prove that this formula is invalid is solve Problem (iii) first. We will see that this formula is equivalent to $\sim q$. It is certainly not a valid formula.

Problem 2 (iii)

Here is a formal proof of the sequent $\vdash \forall p.((p \supset \forall q.q) \supset\sim p)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\vdash p, \sim p} \quad \dfrac{\perp \vdash \sim p}{\forall q.q \vdash \sim p}(*)}{(p \supset \forall q.q) \vdash \sim p}}{\vdash (p \supset \forall q.q) \supset \sim p}}{\vdash \forall p.((p \supset \forall q.q) \supset \sim p)}$$

Remark I wrote this proof starting with the given formula from the bottom to the top.

The most tricky step in the proof is $(*)$. Here we apply the rule

$$\frac{A|^q_B, \Delta \vdash \Gamma}{\forall q.A, \Delta \vdash \Gamma}$$

Here $A = q$. We can chose any arbitrary B. Of course we want to chose B such that the sequent $A|^q_B, \Delta \vdash \Gamma$ would be derivable. Here we take $B = \perp$. Then we get the sequent $\perp, p \vdash$, which is an axiom, since it has $\perp$ as an assumption. Another possible way to prove it is to take $B =\sim p$.

Problem 3 (ii)

$$\forall p.(p \supset q) \supset \forall r.r$$

According to the standard notation the parentheses should be added as follows:

$$(\forall p.(p \supset q)) \supset \forall r.r$$

The wrong solution

$$
\begin{array}{l}
v[(\forall p.(p \supset q)) \supset \forall r.r] = \\
v|_0^p[(p \supset q) \supset \forall r.r] \wedge v|_1^p[(p \supset q) \supset \forall r.r] = \\
v|_0^p[v|_0^r[(p \supset q) \supset r] \wedge v|_0^p[v|_1^r[(p \supset q) \supset r] \wedge \\
v|_1^p[v|_0^r[(p \supset q) \supset r] \wedge v|_1^p[v|_1^r[(p \supset q) \supset r] \wedge
\end{array}
$$

(We use 0 for false and 1 for true.) Remember the definition is

$$v[\forall p.A] = v|_0^p[A] \wedge v|_0^p[A]$$

That is we don't need to copy the entire formula, but only the part of the formula that under the scope of the quantifier.

The above answer would be correct for the following formula:

$$\forall p.\forall r.(p \supset q) \supset r$$

Even if you put parentheses incorrectly: $\forall p.((p \supset q) \supset \forall r.r)$ then the above solution still be wrong, because only r is under the quantifier $\forall r$. So you don't need to copy subformula $p \supset q$ when expand the definition of $v(\forall r.r)$.

The right solution

Note that our formula is of the form $A \supset B$ where $A = \forall p.(p \supset q)$ and $B = \forall r.r$. Let us eliminate quantifiers in A and in B separately.

$$v[B] = v[\forall r.r] = v|_0^r[r] \wedge v|_1^r[r] = 0 \wedge 1$$

$$v[A] = v[\forall p.(p \supset q)] = v|_0^p[p \supset q] \wedge v|_1^p[p \supset q] = (0 \supset v[q]) \wedge (1 \supset v[q])$$

Note that we can simplify this formulas $v[B] = 0 \wedge 1 = 0$, and

$$v[A] = (0 \supset v[q]) \wedge (1 \supset v[q]) = 1 \wedge v[q] = v[q].$$

Therefore the final answer is $v[q] \supset 0 = \sim v[q]$.