Assignment 2, Problem 3

What is the maximum number of nodes that can appear in an analytic tableau (considered as a tree) for a formula of degree n? Explain your answer.

The most common solution was the following:

While going from one level of the tableau to the next the degree of the formula decreases. Therefore the heights of the tableau tree is at most $n+1$. Each node in the tableau tree can have one child (α-type) or two children (β-type). In the worst case if all nodes are β-type we have a binary tree with the heights at most $n+1$. We know that this tree can have at most $2^{n+1}-1$ nodes. Thus we have an upper bound for the number of the nodes in the tableau: $2^{n+1}-1$. This bound is reachable when we have a branch point at every node, i.e. when each node has β-type.

This solution gives us the correct upper bound, but the last sentence is wrong! If all nodes have β-type then the tableau should have only linear number of nodes: $2n+1$.

Indeed, suppose that *all* formulas in the tableau are β. Then we always have to use the last formula in the path at each point. Therefore we use each connectives exactly once. That is, there are exactly n branching points in the tableau tree (forget about negation). We know that a binary tree with n branching point has $2n+1$ nodes. (We can also prove it by induction on the construction of the formula).

Is $2n+1$ correct answer? No. If we alternate α and β formulas we can get exponential many nodes in the tree.

Consider formulas $Y_i = (p_i \wedge q_i)$ and $X_i = Y_1 \vee Y_2 \vee \ldots \vee Y_i$. Take $k = [n/2]$ and construct the tableau for FX_k. First, we get two formulas $F(p_k \wedge q_k)$ and X_{k-1} (lines 2 and 3). Then use the β-formula $F(p_k \wedge q_k)$. This formula gives us two branches (lines 4 and 5). Now unused line (3) lies on the both branches. So we have to use the formula X_{k-1} twice on both branches. Then we will use X_{k-2} four times, and so on.

$$\begin{array}{lll} (1) & FX_k & \\ (2) & Fp_k \wedge q_k & (1) \\ (3) & FX_{k-1} & (1) \end{array}$$

(4)	Fp_k	(2)	(5)	Fq_k	(2)
(6)	$Fp_{k-1} \wedge q_{k-1}$	(3)	(14)	$Fp_{k-1} \wedge q_{k-1}$	(3)
(7)	FX_{k-2}	(3)	(15)	FX_{k-2}	(3)

(8)	Fp_{k-1}	(6)	(9)	Fq_{k-1}	(6)	(16)	Fp_{k-1}	(14)	(17)	Fq_{k-1}	(14)
(10)	$Fp_{k-2} \wedge q_{k-2}$	(7)	(12)	$Fp_{k-2} \wedge q_{k-2}$	(7)	(18)	$Fp_{k-2} \wedge q_{k-2}$	(15)	(20)	$Fp_{k-2} \wedge q_{k-2}$	(15)
(11)	FX_{k-3}	(7)	(13)	FX_{k-3}	(7)	(19)	FX_{k-3}	(15)	(21)	FX_{k-3}	(15)
	$\ldots$			$\ldots$			$\ldots$			$\ldots$	

Let $nodes(X_k)$ be the number of nodes in this tableau. Then $nodes(X_k) = 2nodes(X_{k-1}) + 3$. Therefore $nodes(X_k) = 3 * (2^k - 1)$. Remember that $k =$

$[n/2]$. The degree of X_k is at most n, and the number of nodes in the tableau is $3 * (2^{[n/2]} - 1)$.

Even if we use α-formulas before β-formulas, we still get the exponential number of nodes. In this case we get the tableau with $2^{k+1} + 2k - 1$ nodes:

$$
\begin{array}{lll}
(1) & FX_k & \\
(2) & Fp_k \wedge q_k & (1) \\
(3) & FX_{k-1} & (1) \\
(4) & Fp_{k-1} \wedge q_{k-1} & (3) \\
(5) & FX_{k-2} & (3) \\
 & \ldots & \\
 & \ldots &
\end{array}
$$

$$
\begin{array}{c|c}
\begin{array}{c}
(6) \quad p_k \quad (2) \\
\begin{array}{lll|lll}
(8) & p_{k-1} & (4) & (9) & q_{k-1} & (4) \\
 & \ldots & & & \ldots &
\end{array}
\end{array}
&
\begin{array}{c}
(7) \quad q_k \quad (2) \\
\begin{array}{lll|lll}
(10) & p_{k-1} & (4) & (11) & q_{k-1} & (4) \\
 & \ldots & & & \ldots &
\end{array}
\end{array}
\end{array}
$$