CS 486: Spring 2001
Cornell University
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April 19 Lecture Notes

Summary of past week

We have focused on translating various mathematical concepts into first-order logic, using examples of mathematics from the required textbook by Stewart and Tall, *The Foundations of Mathematics.* I'll abbreviate this as S & T.

We have translated these concepts:

- equivalence relations and equality
- order relations
- algebraic structures

The main translation combined these properties to show how to translate the definition of an inductive ordered integral domain into FOL (first-order logic). This domain characterizes $\mathbb{Z}$, the integers.

Today we will provide a more parsimonious account of numbers in FOL. These are the Peano axioms that define Peano Arithmetic (PA).

PA Axioms

Equality

E1: $x = x$

E2: $x = y \supset (y = z \supset x = z)$

Numbers

N1: $\neg (s(x) = 0)$

N2: $s(x) = s(y) \Leftrightarrow x = y$

N3: $(A(o) \wedge \forall x.(A(x) \supset A(s(x)))) \supset \forall x A(x)$

N4: $(m + 0 = m) \wedge (m + s(n) = s(m + n))$

N5: $(m * 0 = 0) \wedge (m * s(n) = m * n + m)$

With these axioms we can prove the theorems in S & T, p. 151–158.

The above first-order axiomatization provides a bridge between the S & T book and the Smullyan book in the following precise sense. In S & T they establish facts about $\mathbb{N}$ using informal set theory. If these facts are expressible in PA, then they are true in first-order logic and in PA.

Next week we will extend first-order logic to a new logic, called lambda-PRL. The RL stands for Refinement Logic and the P for Program. The first step is to add equality, =, as a primitive. We also add these axioms.

E1: $x = x$

E2: $x = y \supset (A(x) \Leftrightarrow A(y))$

Applying Compactness

Here is a remarkable result of compactness theorem applied to PA. Recall from Smullyan p. 65.

Compactness Theorem. If S is a denumerable set of first-order formulas such that every finite subset is satisfiable, then S is satisfiable.

Let S be a set of true first-order consequences of PA, choosing all particular instances of the axiom schema of induction. Suppose we also add the formulas for a new parameter a_0

$$a_0 \neq 0,\ a_0 \neq 1,\ a_0 \neq 2, \ldots$$

Also add consequences of using a_0 in all theorems. Call this S'.

Every finite subset of S' is satisfiable; we can take a_0 to be a number bigger than any one mentioned. Hence S' is satisfiable. This means that a_0 *behaves just like a natural number,* but it is not any of the natural numbers, $0, 1, 2, 3, \ldots$