CS486 Applied Logic Assignment 9 Due Thurs, April 19, 2001

Reading: Please read *The Foundations of Mathematics*, by Stewart & Tall, to page 80. Most of this is a review.

Exercises:

1. Monoids

   A monoid is an algebraic structure with one operation, op, which is *associative* and has a unit element, 1, such that

   $$x \text{ op } 1 = x \quad \text{and} \quad 1 \text{ op } x = x.$$

   Say that a monoid is *commutative* iff x op $y = y$ op x, and is *idempotent* iff x op $x = x$.

   (a) Write the axioms for a commutative idempotent monoid using first-order logic.

   Prove or disprove using first-order logic that:

   (b) The unit element of a monoid is unique.

   (c) An idempotent monoid is commutative.

2. Boolean Rings

   (a) A Boolean ring R is a ring (with unit) such that $x * x = x$ for all $x \in R$. Show that any Boolean ring is commutative, i.e., $x * y = y * x$ for all $x, y \in R$.

   (b) Define $x + y$ as $x \Leftrightarrow y$, $-x$ as x, and $x * y$ as $x \vee y$.
   Take 0 as true (T) and 1 as false ($\perp$).

   Show that with these definitions the class of propositions, *Prop*, is a Boolean ring. Use the truth definition to prove properties of the logical connectives and constants.

   Note, you need not express this in first-order logic.

3. Problem 7, page 82, in Stuart & Tall.