1 metavariables

$p, q, r, \ldots$	a propositional variable
$A, B, \ldots$	a $\mathbf{P^2}$ formula
$\Gamma, \Delta, \ldots$	a finite set of $\mathbf{P^2}$ formulas

2 syntax of $\mathbf{P^2}$

The formulas of $\mathbf{P^2}$ are generated by

$$\begin{array}{lcl} V & \to & p_0 \mid p_1 \mid p_2 \mid \cdots \mid p_i \mid \cdots \qquad \text{(a countably infinite set)} \\ A & \to & V \mid \bot \mid (A \supset A') \mid (\forall V A) \end{array}$$

examples: $(p_0 \supset p_1)$, $(\forall p_0 (p_0 \supset p_1))$, $(\forall p_1 ((\forall p_2 (p_2 \supset p_2)) \supset \bot))$

The remaining connectives and quantifier can be defined in terms of $\bot$, $\supset$, and $\forall$:

$$\begin{array}{rcl} \neg A & \leftrightarrow & A \supset \bot \\ A \wedge B & \leftrightarrow & \neg(A \supset \neg B) \\ A \vee B & \leftrightarrow & (\neg A) \supset B \\ \exists p A & \leftrightarrow & \neg \forall p \neg A \end{array}$$

3 assignments

Let *Var* be the type of propositional variables, and let $\mathbb{B} = \{0, 1\}$ be the booleans (with 0 meaning false and 1 meaning true). An assignment is a function $v : \mathit{Var} \to \mathbb{B}$.

Given an assignment v, a boolean b, and a propositional variable p, the "updated" assignment $v|^p_b$ is the function (in $\mathit{Var} \to \mathbb{B}$) defined by

$$(v|^p_b)(q) = \begin{cases} b & \text{if } q = p \\ v(q) & \text{o.w.} \end{cases}$$

4 semantics of $\mathbf{P^2}$

Let A be a $\mathbf{P^2}$-formula and let v be an assignment; let $v[A]$ be the notation for the (boolean) value of A under v, and let $v[A] : \mathbb{B}$ be defined recursively as follows:

$$\begin{array}{lcl} v[\bot] & = & 0 \\ v[p] & = & v(p) \\ v[A \supset B] & = & (\neg_{\mathbb{B}}\, v[A]) \ \vee_{\mathbb{B}} \ v[B] \\ v[\forall p A] & = & (v|^p_0)[A] \ \wedge_{\mathbb{B}} \ (v|^p_1)[A] \end{array}$$

where $\neg_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$, $\vee_{\mathbb{B}} : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$, and $\wedge_{\mathbb{B}} : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ are the standard boolean operators.

For a finite set of formulas Γ, define $v_\wedge[\Delta] = \bigwedge_{\mathbb{B}} \{ v[A] \mid A \in \Delta \}$ and define $v_\vee[\Gamma] = \bigvee_{\mathbb{B}} \{ v[A] \mid A \in \Gamma \}$, where $\bigwedge_{\mathbb{B}} S$ is the conjunction of the boolean values in the set S and $\bigvee_{\mathbb{B}} S$ is their disjunction. (By convention, $\bigwedge_{\mathbb{B}} \emptyset = 1$ and $\bigvee_{\mathbb{B}} \emptyset = 0$.) The value $v[\Delta \vdash \Gamma]$ of a sequent can now be defined as $(\neg_{\mathbb{B}}\, v_\wedge[\Delta]) \ \vee_{\mathbb{B}} \ v_\vee[\Gamma]$.

5 free variables

For A a formula of $\mathbf{P^2}$, the set of propositional variables that are free in A, denoted $FV(A)$, can be characterized by the following recursive definition:

$$\begin{array}{lcl} FV(\bot) & = & \emptyset \\ FV(p) & = & \{p\} \\ FV(A \supset B) & = & FV(A) \cup FV(B) \\ FV(\forall p A) & = & FV(A) - \{p\} \end{array}$$

The set of all propositional variables that occur in A, $PV(A)$, can likewise be defined as

$$
\begin{array}{lcl}
PV(\bot) & = & \emptyset \\
PV(p) & = & \{p\} \\
PV(A \supset B) & = & PV(A) \cup PV(B) \\
PV(\forall p A) & = & PV(A) \cup \{p\}
\end{array}
$$

examples:

$$
\begin{array}{lcl}
FV(p_0 \supset p_1) & = & \{p_0, p_1\} \\
PV(p_0 \supset p_1) & = & \{p_0, p_1\} \\
FV(\forall p_0(p_0 \supset p_1)) & = & \{p_1\} \\
PV(\forall p_0(p_0 \supset p_1)) & = & \{p_0, p_1\} \\
FV(\forall p_1((\forall p_2(p_2 \supset p_2)) \supset \bot)) & = & \emptyset \\
PV(\forall p_1((\forall p_2(p_2 \supset p_2)) \supset (\forall p_3 p_1))) & = & \{p_1, p_2, p_3\}
\end{array}
$$

Extend the definitions of FV and PV to finite sets of formulas by taking $FV(\Gamma) = \bigcup_{A \in \Gamma} FV(A)$ and likewise by taking $PV(\Gamma) = \bigcup_{A \in \Gamma} PV(A)$. For sequents, the definitions are $FV(\Delta \vdash \Gamma) = FV(\Delta \cup \Gamma)$ and $PV(\Delta \vdash \Gamma) = PV(\Delta \cup \Gamma)$.

6 substitution

Given formulas A and B of $\mathbf{P^2}$ and a propositional variable p, the $\mathbf{P^2}$ formula $A|^p_B$ ("A with B substituted for p") is, as usual, defined recursively:

$$
\begin{array}{rcll}
\bot|^p_B & = & \bot & \\
p|^p_B & = & B & \\
q|^p_B & = & q & (q \neq p) \\
(A \supset A')|^p_B & = & (A|^p_B) \supset (A'|^p_B) & \\
(\forall p A)|^p_B & = & \forall p A & \\
(\forall q A)|^p_B & = & \forall q(A|^p_B) & (q \neq p,\ q \notin FV(B)) \\
(\forall q A)|^p_B & = & \forall q'(A|^q_{q'}|^p_B) & (q \neq p,\ q \in FV(B),\ q' \notin PV(A, B, p))
\end{array}
$$

examples:

$$
\begin{array}{rcl}
(p_0 \supset p_1)|^{p_0}_{p_2 \supset p_3} & = & ((p_2 \supset p_3) \supset p_1) \\
(p_0 \supset (p_0 \supset p_1))|^{p_0}_{p_3} & = & (p_3 \supset (p_3 \supset p_1)) \\
(p_0 \supset p_0)|^{p_0}_{p_0 \supset p_0} & = & ((p_0 \supset p_0) \supset (p_0 \supset p_0)) \\
(p_0 \supset (\forall p_0(p_0 \supset p_0)))|^{p_0}_{p_1} & = & (p_1 \supset (\forall p_0(p_0 \supset p_0))) \\
(\forall p_0(p_0 \supset p_3))|^{p_3}_{p_0} & = & (\forall p_1(p_1 \supset p_0))
\end{array}
$$

One can extend substitution to finite sets of formulas and thence to sequents by letting $\Gamma|^p_B = \{\, A|^p_B \mid A \in \Gamma \,\}$ and $(\Delta \vdash \Gamma)|^p_B = (\Delta|^p_B) \vdash (\Gamma|^p_B)$.

7 rules of $\mathbf{P^2}$

The multiple-conclusioned sequent proof rules for $\mathbf{P^2}$ are (in "root-down tree format")

$$\Delta, \bot \vdash \Gamma$$

$$\frac{\Delta \vdash A, \Gamma \qquad \Delta, B \vdash \Gamma}{\Delta, A \supset B \vdash \Gamma} \qquad \frac{\Delta, A \vdash B, \Gamma}{\Delta \vdash A \supset B, \Gamma}$$

$$\frac{\Delta, \forall pA, A|^p_B \vdash \Gamma}{\Delta, \forall pA \vdash \Gamma} \qquad \frac{\Delta \vdash A|^p_q, \Gamma}{\Delta \vdash \forall pA, \Gamma}\ (!)$$

$$\Delta, A \vdash A, \Gamma$$

$$\frac{\Delta \vdash \Gamma}{\Delta, A \vdash \Gamma} \qquad \frac{\Delta \vdash \Gamma}{\Delta \vdash A, \Gamma}$$

(!) this is only legal if $q \notin FV(\Delta, \Gamma, \forall pA)$.

The rules for $\exists$ can be derived from the rules given above:

$$\frac{\Delta, A|^p_q \vdash \Gamma}{\Delta, \exists pA \vdash \Gamma}\ (!) \qquad \frac{\Delta \vdash A|^p_B, \Gamma}{\Delta \vdash \exists pA, \Gamma}$$

The familiar rules for $\wedge$, $\vee$, and $\neg$ can also be derived.

an example proof:

$$\cfrac{\cfrac{\bot \vdash \bot}{\forall p.p \vdash \bot}}{\vdash (\forall p.p) \supset \bot}$$

The topmost step is the left $\forall$ rule, using $\bot$ as B; i.e., informally, the proof is

$$\cfrac{\cfrac{\cfrac{\bot \vdash \bot}{p|^p_\bot \vdash \bot}}{\forall p.p \vdash \bot}}{\vdash (\forall p.p) \supset \bot}$$

where the topmost pseudo-step is justified (meta-theoretically) by the equality $p|^p_\bot = \bot$.