1 metavarniables
$p, q, r, \ldots$  a propositional variable
$A, B, \ldots$  a $P^2$ formula
$\Gamma, \Delta, \ldots$  a finite set of $P^2$ formulas

2 syntax of $P^2$
The formulas of $P^2$ are generated by

\[ V \rightarrow p_0 \mid p_1 \mid p_2 \mid \cdots \mid p_i \mid \cdots \quad (\text{a countably infinite set}) \]
\[ A \rightarrow V \mid \bot \mid (A \supset A') \mid (\forall A) \]

examples:  $(p_0 \supset p_1), (\forall p_0 (p_0 \supset p_1)), (\forall p_1 ((\forall p_2 (p_0 \supset p_2)) \supset \bot))$

The remaining connectives and quantifiers can be defined in terms of $\bot, \supset, \forall$:

\[ \neg A \leftrightarrow A \supset \bot \]
\[ A \land B \leftrightarrow (\neg A \supset \neg B) \]
\[ A \lor B \leftrightarrow (\neg A \supset B) \]
\[ \exists p A \leftrightarrow \forall p \neg A \]

3 assignments
Let $\text{Var}$ be the type of propositional variables, and let $\mathbb{B} = \{0, 1\}$ be the booleans (with 0 meaning false and 1 meaning true). An assignment is a function $v : \text{Var} \to \mathbb{B}$.

Given an assignment $v$, a boolean $b$, and a propositional variable $p$, the “updated” assignment $v[p]_b$ is the function (in $\text{Var} \to \mathbb{B}$) defined by

\[ (v[p]_b)(q) = \begin{cases} b & \text{if } q = p \\ v(q) & \text{o.w.} \end{cases} \]

4 semantics of $P^2$
Let $A$ be a $P^2$-formula and let $v$ be an assignment; let $v[A]$ be the notation for the (boolean) value of $A$ under $v$, and let $v[A] : \mathbb{B}$ be defined recursively as follows:

\[ v[\bot] = 0 \]
\[ v[p] = v(p) \]
\[ v[A \supset B] = (\neg v[A]) \lor v[B] \]
\[ v[\forall p A] = (v[p])_v[A] \land v[A] \]

where $\neg : \mathbb{B} \to \mathbb{B}$, $\lor : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$, and $\land : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ are the standard boolean operators.

For a finite set of formulas $\Gamma$, define $v_{\land} [\Delta] = \land_{\mathbb{B}} \{ v[A] \mid A \in \Delta \}$ and define $v_{\lor} [\Gamma] = \lor_{\mathbb{B}} \{ v[A] \mid A \in \Gamma \}$, where $\land_{\mathbb{B}} S$ is the conjunction of the boolean values in the set $S$ and $\lor_{\mathbb{B}} S$ is their disjunction. (By convention, $\land_{\mathbb{B}} \emptyset = 1$ and $\lor_{\mathbb{B}} \emptyset = 0$.) The value $v[\Delta \vdash \Gamma]$ of a sequent can now be defined as $\neg v_{\land} [\Delta] \lor v_{\lor} [\Gamma]$.

5 free variables
For $A$ a formula of $P^2$, the set of propositional variables that are free in $A$, denoted $FV(A)$, can be characterized by the following recursive definition:

\[ FV(\bot) = \emptyset \]
\[ FV(p) = \{ p \} \]
\[ FV(A \supset B) = FV(A) \cup FV(B) \]
\[ FV(\forall p A) = FV(A) - \{ p \} \]
The set of all propositional variables that occur in \( A \), \( PV(A) \), can likewise be defined as

\[
\begin{align*}
PV(\bot) &= \emptyset \\
PV(p) &= \{p\} \\
PV(A \supset B) &= PV(A) \cup PV(B) \\
PV(\forall pA) &= PV(A) \cup \{p\}
\end{align*}
\]

examples:

\[
\begin{align*}
FV(p_0 \supset p_1) &= \{p_0, p_1\} \\
FV(p_0 \supset p_1) &= \{p_0, p_1\} \\
FV(\forall p_0((p_0 \supset p_1))) &= \{p_1\} \\
FV(\forall p_0((p_0 \supset p_1))) &= \{p_0, p_1\} \\
FV(\forall p_1((\forall p_2(p_2 \supset p_2) \supset \bot))) &= 0 \\
FV(\forall p_1((\forall p_2(p_2 \supset p_2) \supset (\forall p_3 p_1))) &= \{p_1, p_2, p_3\}
\end{align*}
\]

Extend the definitions of \( FV \) and \( PV \) to finite sets of formulas by taking \( FV(\Gamma) = \bigcup_{A \in \Gamma} PV(A) \) and likewise by taking \( PV(\Gamma) = \bigcup_{A \in \Gamma} PV(A) \). For sequents, the definitions are \( FV(\Delta \vdash \Gamma) = FV(\Delta \cup \Gamma) \) and \( PV(\Delta \vdash \Gamma) = PV(\Delta \cup \Gamma) \).

6 substitution

Given formulas \( A \) and \( B \) of \( \mathbf{P}^2 \) and a propositional variable \( p \), the \( \mathbf{P}^2 \) formula \( A|_p^p \) ("\( A \) with \( B \) substituted for \( p' \)) is, as usual, defined recursively:

\[
\begin{align*}
\bot|_p^p &= \bot \\
p|_p^p &= B \\
qu|_p^p &= q \quad (q \neq p) \\
(A \supset A')|_p^p &= (A|_p^p \supset (A'|_p^p)) \\
(\forall p A)|_p^p &= \forall p A \\
(\forall q A)|_p^p &= \forall q(A|_q^q) \quad (q \neq p, q \notin FV(B)) \\
(\forall q A)|_p^p &= \forall q'(A|_{q'}^q) \quad (q \neq p, q \notin FV(B), q' \notin PV(A, B, p))
\end{align*}
\]

examples:

\[
\begin{align*}
(p_0 \supset p_1)|_{p_0 \supset p_0}^{p_0} &= ((p_2 \supset p_3) \supset p_1) \\
(p_0 \supset (p_0 \supset p_1))|_{p_0}^{p_0} &= (p_2 \supset (p_3 \supset p_1)) \\
(p_0 \supset p_0)|_{p_0 \supset p_0}^{p_0} &= ((p_0 \supset p_0) \supset (p_0 \supset p_0)) \\
(p_0 \supset (\forall p_1 (p_0 \supset p_0))|_{p_1}^{p_0} &= (p_1 \supset (\forall p_0 (p_0 \supset p_0))) \\
(\forall p_0 (p_0 \supset p_0))|_{p_0}^{p_0} &= (\forall p_1 (p_0 \supset p_0))
\end{align*}
\]

One can extend substitution to finite sets of formulas and hence to sequents by letting \( \Gamma|_p^p = \{ A|_p^p \mid A \in \Gamma \} \) and \( (\Delta \vdash \Gamma)|_p^p = (\Delta|_p^p \vdash (\Gamma|_p^p) \).

2
7 rules of $\mathbb{P}^2$

The multiple-conclusioned sequent proof rules for $\mathbb{P}^2$ are (in “root-down tree format”)

$$
\Delta, \bot \vdash \Gamma
$$

\[ \begin{array}{ll}
\Delta \vdash A, \Gamma & \Delta, A \supset B \vdash \Gamma \\
\Delta, A \supset B, \Gamma & \Delta \vdash A \supset B, \Gamma \\
\Delta, \forall p. A \vdash \Gamma & \Delta \vdash \forall p. A, \Gamma \quad (!)
\end{array} \]

\[ \begin{array}{ll}
\Delta \vdash A, \Gamma & \Delta \vdash \Gamma \\
\Delta, A \vdash \Gamma & \Delta \vdash A, \Gamma
\end{array} \]

(!) this is only legal if $q \notin FV(\Delta, \forall p. A)$.

The rules for $\exists$ can be derived from the rules given above:

\[ \begin{array}{ll}
\frac{\Delta, A \vdash \Gamma}{\Delta, \exists p. A \vdash \Gamma} \quad (!) & \frac{\Delta \vdash A, \Gamma}{\Delta \vdash \exists p. A, \Gamma}
\end{array} \]

The familiar rules for $\land$, $\lor$, and $\neg$ can also be derived.

an example proof:

\[ \frac{\bot \vdash \bot}{\forall p. p \vdash \bot} \frac{\bot \vdash \bot}{\vdash (\forall p. p) \supset \bot} \]

The topmost step is the left $\forall$ rule, using $\bot$ as $B$; i.e., informally, the proof is

\[ \frac{\bot \vdash \bot}{\bot \vdash \bot} \frac{\bot \vdash \bot}{\forall p. p \vdash \bot} \frac{\bot \vdash \bot}{\vdash (\forall p. p) \supset \bot} \]

where the topmost pseudo-step is justified (meta-theoretically) by the equality $p|_{\bot}^\circ = \bot$. 

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