$\boxed{\text{6 May 2022}}$ Bounding Mixing Times

ε-mixing time τ means: for any initial distrib π_0, for all $t \geq \tau$,

$$\| \pi_t - \pi \|_{TV} \leq \varepsilon$$

where π denotes stationary distribution, π_t denotes time t distribution.

Coupling with transition matrix P and initial distributions π_0, π_0' is a sequence of pairs $\{(X_t, X'_t)\}_{t=0,1,2,\ldots}$ s.t.

$X_0, X_1, \ldots \longrightarrow$ is a Markov chain with trans mtx P

$X_0', X_1', \ldots \longrightarrow$

$X_0 \sim \pi_0 \qquad X_0' \sim \pi_0'$

Markov Coupling Lemma: For transition matrix P, to prove ε-mix time is $\leq \tau$, it suffices to show for all initial state distribs π_0 $\exists$ a Markov coupling with transition mtx P and initial distribs π_0, π, such that

$$\forall t \geq \tau \quad \Pr(X_t \neq X_t') \leq \varepsilon.$$

Glauber dynamics: Markov chain on proper q-colorings of graph $G=(V,E)$ defined by this state transition dynamics: in state $x: V \to [q]$

- choose $v \in V$, $c \in [q]$ uniformly at random
- let $y: V \to [q]$ be defined as

$$y(u) = \begin{cases} c & \text{if } u = v \\ x(u) & \text{if } u \neq v \end{cases}$$

- if y is a proper coloring transition to y else remain at x.

Assume max degree of G is Δ, $q > 4\Delta$.

Couple X_t, X'_t by proposing to recolor <u>same</u> <u>vertex</u> v_t with <u>same</u> <u>color</u> c_t in both Markov chains.

Analyze $\Pr(X_t \neq X'_t)$ using Hamming distance

$$d_t = \#\{u \mid X_t(u) \neq X'_t(u)\}$$

Observe $d_t \geq 1$ when $X_t \neq X'_t$ so $\Pr(X_t \neq X'_t) \leq \mathbb{E}[d_t]$.

How does d_{t+1} differ from d_t?

$$d_t - d_{t+1} \in \{-1, 0, 1\}.$$

(v_t is the only vertex whose color changed)

$d_{t+1} = d_t - 1$ when a <u>color</u> <u>merge</u> takes place.

$X_t(v_t) \neq X'_t(v_t)$ but

$X_{t+1}(v_t) = X'_{t+1}(v_t) = c_t$

Color merge happens exactly when — Prob

- v_t is one of the d_t vertices colored differently in X_t, X'_t — $\frac{d_t}{n}$
- c_t is not one of the neighbors' colors in X_t or X'_t. — $\geq \frac{q-2\Delta}{q}$

$\Pr(\text{color merge}) \geq \left(\frac{q-2\Delta}{q}\right)\left(\frac{d_t}{n}\right)$

$d_{t+1} = d_t + 1$ when a color split occurs.

- $X_t(v_t) = X'_t(v_t)$
- $X_{t+1}(v_t) = c_t$, $X'_{t+1}(v_t) \neq c_t$ or vice-versa

because $\exists$ neighbor w s.t. $X_t(w) \neq c_t$, $X'_t(w) = c_t$

or $X_t(w) = c_t$, $X'_t(w) \neq c_t$.

Blame the split on directed edge (v_t, w).

Every color split can be blamed on at least one edge.

$\Pr(\text{color split}) \leq \mathbb{E}[\text{blamed edges}]$

$= \sum_{(v,w)} \underbrace{\Pr((v,w) \text{ is blamed for a color split})}$

$= 0$ unless w is one of the d_t vertices where $X_t \neq X'_t$ and $c_t \in \{X_t(w), X'_t(w)\}$ and v_t is neighbor of w.

$\leq d_t \cdot \Delta \cdot 2 \cdot \frac{1}{nq}$

$= \frac{d_t}{n} \cdot \frac{2\Delta}{q}$

$$\Pr(\text{color merge}) \geq \left(\frac{q-2\Delta}{nq}\right) d_t$$

$$\Pr(\text{color split}) \leq \left(\frac{2\Delta}{nq}\right) d_t$$

$$E[d_{t+1} \mid d_t] = d_t - \Pr(\text{color merge}) + \Pr(\text{color split})$$

$$\leq d_t - \left(\frac{q-2\Delta}{nq}\right) d_t + \left(\frac{2\Delta}{nq}\right) d_t$$

$$= \left(1 - \frac{q-4\Delta}{nq}\right) d_t$$

Induct on t:

$$E[d_t] \leq \left(1 - \frac{q-4\Delta}{nq}\right)^t d_0$$

$$\leq \left(1 - \frac{q-4\Delta}{nq}\right)^t n$$

$$< \exp\left(-\left(\frac{q-4\Delta}{nq}\right)t + \ln(n)\right)$$

To make this $\leq \varepsilon$, we need

$$\exp\left(-\left(\frac{q-4\Delta}{nq}\right)t + \ln(n)\right) \leq \varepsilon$$

$$\left(\frac{q-4\Delta}{nq}\right)t - \ln(n) \geq \ln\left(\frac{1}{\varepsilon}\right)$$

$$t \geq \boxed{\left(\frac{q}{q-4\Delta}\right) n \ln\left(\frac{n}{\varepsilon}\right)}$$

mixing time upper bound.