15 Apr 2022

Sparse Recovery.

Solving $Rx = b$ where $x \in \mathbb{R}^n$, s-sparse

$b \in \mathbb{R}^k$ $k < n$

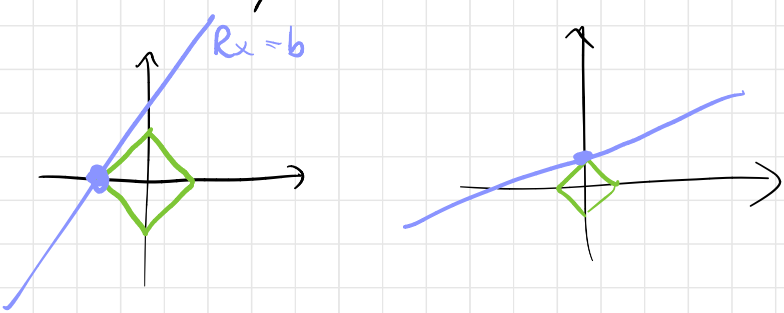
R has $(3s)$-restricted isometry property with coefficient ε.

$\forall\, z \in \mathbb{R}^n$, z $(3s)$-sparse

$$(1-\varepsilon)\|z\|_2^2 \leq \|Rz\|_2^2 \leq (1+\varepsilon)\|z\|_2^2$$

<u>Algorithm</u>. Among all x that satisfy $Rx = b$ choose one with minimum 1-norm.

<u>Picture</u>. Say $n = 2$, $k = 1$, $R: \mathbb{R}^2 \to \mathbb{R}^1$.



$\|x\|_1$ is a convex function of x.

Minimize $\|x\|_1$ using gradient descent. (or some other convex opt. procedure)

A commonly used notation $\|x\|_0$ denotes the # of non-zero coordinates of x.

We minimize $\|x\|_1$ rather than $\|x\|_0$ because we have an efficient algorithm for the $\|\cdot\|_1$ objective but not the $\|\cdot\|_0$.

<u>Def.</u> Vector $z \in \mathbb{R}^n$ is <u>mostly</u> <u>s-sparse</u> if $\exists J \subseteq [n]$, $|J| = s$, s.t.

$$\sum_{i \in J} |z_i| > \sum_{i \notin J} |z_i|$$

<u>Lem.</u> If R satisfies $(3s)$-RIP with const. $\varepsilon < 3 - 2\sqrt{2} \approx 0.17\ldots$ and z is mostly s-sparse then

$$\|Rz\|_2 \geq \left(1 - \varepsilon - \frac{1+\varepsilon}{\sqrt{2}}\right) \|z\|_2$$

<u>Proof.</u> (To be presented if time permits...)

Finishing up analysis of L_1 minimization.

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Suppose x_0 is s-sparse and x_1 is the output of

$$\min \|x\|_1 \quad \text{s.t. } Rx = b = Rx_0$$

Then $z = x_1 - x_0$ satisfies $Rz = R(x_1 - x_0) = 0$.

We know $\|x_1\|_1 \leq \|x_0\|_1 = \sum_{i\in J} |x_{0i}|$

$J = \{\text{nonzero coords of } x_0\}$

$$\|x_1\|_1 = \sum_{i \in J} |x_{0i} + z_i| + \sum_{i \notin J} |z_i|$$

$$\geq \sum_{i\in J} |x_{0i}| - \sum_{i \in J} |z_i| + \sum_{i \notin J} |z_i|$$

$$\therefore \sum_{i\in J} |z_i| \geq \sum_{i \notin J} |z_i|$$

$\therefore$ z is mostly s-sparse.

$\therefore$ (by Lemma) $z = 0 \Rightarrow x_0 = x_1$.