$\boxed{\text{9 Mar 2022}}$ Gaussians II

The multivariate normal distribution $N(0, \mathbb{1})$ satisfies:

1. If $X = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ is sampled from $N(0, \mathbb{1})$ then $X_1, X_2, \dots, X_n$ are indep. and each is $N(0,1)$.

2. If Q is an $n \times n$ orthogonal matrix and $X \sim N(0, \mathbb{1})$ then $QX \sim N(0, \mathbb{1})$.

Ex. If X_1, X_2 are indep $N(0,1)$, what's the distribution of $X_1 + 2X_2$?

Hypothesis: $X_1 + 2X_2$ is $N(0,5)$.

$$\iff \underbrace{\frac{1}{\sqrt{5}} X_1 + \frac{2}{\sqrt{5}} X_2}_{= \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}} \text{ is } N(0,1).$$

$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ is orthog.

$$\therefore \quad Q\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ is } N(0,\mathbb{1})$$

$$\therefore \quad \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ is } N(0,1).$$

$$\therefore \quad \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ is } N(0,5).$$

<u>Ex. 2.</u> If $X_1, \dots, X_n$ are $\underbrace{\text{i.i.d.}}_{}$ $N(0,1)$ rv's

independent and identically distributed

$$\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) \text{ is also } N(0,1)$$

because $\underbrace{\begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix}}_{n \text{ dimensional}}$

is the first row of an orthogonal $n \times n$ matrix.

For a random vector $X \in \mathbb{R}^d$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_d \end{bmatrix}$$

$$\operatorname{Cov}(X_i, X_j) \triangleq E[X_i X_j] - E[X_i]\,E[X_j].$$

for a vector $w = \begin{bmatrix} w_1 \\ \vdots \\ w_d \end{bmatrix}$, the matrix ww^T has the (i,j) entry $w_i w_j$.

Def. The covariance matrix of X is

$$\mathrm{Cov}(X) \triangleq \mathbb{E}[XX^T] - (\mathbb{E}X)(\mathbb{E}X)^T$$

(If Y is rv taking values in vect spc V with density $f(y)$.

$$\mathbb{E}[Y] = \int_V y \, f(y) \, dy \quad)$$

Covariance of $N(0, \mathbb{1})$. If $X \sim N(0, \mathbb{1})$

$$\mathbb{E}[X_i X_j] = \begin{cases} 1 & \text{if } i = j \\ \underset{\displaystyle 0}{\mathbb{E}[X_i]\mathbb{E}[X_j]} & \text{if } i \neq j \end{cases}$$

Now what if $X \in \mathbb{R}^d$ is distributed as $N(0, \mathbb{1})$ and $Y = BX + \mu$ for some invertible $d \times d$ B and $\vec{\mu} \in \mathbb{R}^d$.

linearity of expectation

$$\mathbb{E}[Y] = \mathbb{E}[BX] + \mu = B\,\underbrace{\cancel{\mathbb{E}[X]}}_{0} + \mu = \mu$$

$$\mathbb{E}[YY^T] = \mathbb{E}\left[(BX+\mu)(X^TB^T+\mu^T)\right]$$

$$= \mathbb{E}[BXX^TB^T] + \mathbb{E}[BX\mu^T] + \mathbb{E}[\mu X^TB^T] + \mu\mu^T$$

$$= B\,\mathbb{E}[\cancelto{\mathbb{1}}{XX^T}]B^T + B\,\mathbb{E}[\cancelto{0}{X}]\mu^T + \mu\,\mathbb{E}[\cancelto{0}{X^T}]B^T + \mu\mu^T$$

$$= BB^T + \mu\mu^T$$

$$\mathrm{Cov}(Y) = BB^T.$$

Conclusion: if $X \sim N(0, \mathbb{1})$ and $Y = BX + \mu$ then $Y \sim N(\mu, BB^T)$.

FACT. If two Gaussian distributions have same mean and covariance, they are equal!

Suppose $X \in \mathbb{R}^d$ is $N(0, \mathbb{1})$ and $Y = AX$ where A is $d \times n$, rank d.

Claim: Y has a Gaussian distribution

Proof. Use SVD! $A = USV^T$, U, V orthogonal,

$$S = D \cdot \left[\mathbb{1}_{d\times d}\ 0_{d\times(n-d)}\right] = \begin{bmatrix} \begin{matrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_d \end{matrix} & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_d \end{bmatrix}$$

$$Y = UD\,[\mathbb{1}\ 0]\,V^T X$$

$$\sim UD\,[\mathbb{1}\,0]\,X$$ b/c V^T is orthogonal, X is rot-inv't

$$\sim UD\,X_{1:d} \quad \sim \mathcal{N}(0, \mathbb{1}_{d \times d})$$

$$\sim BX_{1:d}$$ where $B = UD$ invertible

So Y Gaussian $\mathcal{N}(0, BB^T)$

$$= \mathcal{N}(0, UD^2U^T).$$