Continuous distributions and Gaussians

(Reading: §12.4)

Announcements: Quiz 1 grades to be released tonight.

On Gradescope/CMS (total grade)
(Make-up quiz yet to be graded)

Without "drop lowest score" correction, here's how things look...

24+ = perfect score
17+ $\rightarrow$ A +/-
13-17 $\rightarrow$ B +/-
10-13 $\rightarrow$ C +/-
<10 $\rightarrow$ you're probably omitting this quiz score.

We say random variables $X, X' \in \mathbb{R}^d$ have the same distribution if

\[ \forall \text{ measurable set } S \subseteq \mathbb{R}^d \quad \Pr(X \in S) = \Pr(X' \in S). \]

To say that random variable $X \in \mathbb{R}^d$ has probability density $f$ means

\[ \forall S \quad \Pr(X \in S) = \int_S f(y) \, dy \]

Density always satisfies $\int_{\mathbb{R}^d} f(y) \, dy = 1$, by $f(y) \geq 0$. 
Fact: If \( X, X' \) have the same density, they have the same distribution.

\[ \text{Not every density has a density, e.g. } X = \delta_0 \text{ w/ probability } 1. \]

When \( X \) has a density, the density \( f(y) \) can be defined as

\[
f(y) = \lim_{\text{diam}(S) \to 0} \frac{\Pr(X \in S)}{\text{Vol}(S)}, \quad \forall y \in S
\]

For random variable \( X \in \mathbb{R}^2 \), the cumulative distribution function (CDF) is \( F \) defined by

\[
F(\theta) = \Pr(X \in (-\infty, \theta]) = \Pr(X \leq \theta)
\]

Useful fact: If \( X \) has a continuous CDF, \( F \), then the random variable \( F(X) \) is uniformly distributed in \([0,1]\).

Distribution of \( Y = F(X) \), satisfies \( \Pr(Y < \theta) = \Pr(F(X) < \theta) \) \( \forall \theta \).

"\( Z \) is unit on \([0,1]\)" means

\[
\forall 0 \leq a \leq b \leq 1, \quad \Pr(Z \in [a,b]) = b - a.
\]
If $F$ is the (continuous) CDF of $X$ and $Y = F(X)$, then why is $Y$ un

If $F$ is continuous and $F(x)$ is unit $[0,1]$ then $F$ is the CDF of $X$

$\Rightarrow$ if $Y$ is unif $[0,1]$ and $F$ is continuous, strictly increasing then $X = F^{-1}(Y)$ has CDF $F$

Ex. Exponential distribution with rate $r$

is characterized by

$Pr(X > \theta) = e^{-r\theta}$

$F(\theta) = 1 - e^{-r\theta}$

$F^{-1}(y) =$ the $\theta$ that solves

$1 - e^{-r\theta} = y$

$\theta = \frac{1}{r} \ln \left( \frac{1}{1-y} \right)$. 