$\boxed{\text{14 Feb 2022}}$ Change of basis formulae, starting SVD

Recall course enrollment example:

$$M = \begin{array}{cc} & \text{course} \\ \#\text{ ugrad} & 50 \\ \#\text{ grad} & 20 \end{array} \qquad M' = \begin{array}{cc} & \text{course} \\ \text{total \# stu} & 70 \\ \#\text{ grad} & 20 \end{array}$$

Change of basis matrix $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. $M' = BM$

But if a linear function has coefficient vector $[2 \;\; 1]$ in first basis, its coefficient vector in the second one is $[2 \;\; -1]$.

Def. A based vector space is V with isomorphism $\mathbb{R}^n \xrightarrow{\beta} V$.

Choosing a based ves. struct on V lets us write elements of V as n-tuples of numbers
$(v) \qquad\qquad \beta^{-1}(v)$.

Example. V = vector space of course enrollments.

$\beta_1 : \mathbb{R}^2 \to V$ basis choice for table M.

$\beta_1\begin{pmatrix}1\\0\end{pmatrix}$ = a vector of course enrolls that would be rep'd in M by column vector $\begin{pmatrix}1\\0\end{pmatrix}$.

= a course with 1 ugrad, $\emptyset$ grad

$\triangleq \vec{u}$.

$\beta_1\begin{pmatrix}0\\1\end{pmatrix} \triangleq \vec{g}$ = course with 1 grad, $\emptyset$ ugrad

$\beta_2 : \mathbb{R}^2 \to V$ basis choice for table M'

$\beta_2\begin{pmatrix}1\\0\end{pmatrix}$ = enrollment represented in M' by $\begin{pmatrix}1\\0\end{pmatrix}$ i.e. 1 total student, $\emptyset$ grads

$= \vec{u}$

$\beta_2\begin{pmatrix}0\\1\end{pmatrix}$ = enroll rep'd in M' by $\begin{pmatrix}0\\1\end{pmatrix}$ i.e. $\emptyset$ total students, 1 of whom is grad.

$= \vec{g} - \vec{u}$

<u>Def.</u> Change of basis matrix from β_1 to β_2 is the matrix representing the linear transformation

$$\mathbb{R}^n \xrightarrow{\beta_1} V \xrightarrow{\beta_2^{-1}} \mathbb{R}^n \qquad \beta_2^{-1} \circ \beta_1$$

$$\beta_2^{-1}(\beta_1(\vec{e_1})) = \beta_2^{-1}(\vec{u}) = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\beta_2^{-1}(\beta_1(\vec{e_2})) = \beta_2^{-1}(\vec{g}) \qquad B = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}.$$

$$= \beta_2^{-1}(\vec{u} + \vec{g} - \vec{u}) = \begin{pmatrix}1\\1\end{pmatrix}$$

What does change-of-basis do to the matrix representing a linear transformation, $T: V \to W$?

$$\begin{tikzcd}
 & V \arrow[rr, "T"] & & W & \\
\mathbb{R}^n \arrow[ur, "\beta_{V1}"] & \mathbb{R}^n \arrow[u, "\beta_{V2}"'] & \mathbb{R}^m \arrow[ur, "\beta_{W1}"] & & \mathbb{R}^m \arrow[ul, "\beta_{W2}"']
\end{tikzcd}$$

To say that M_1 represents T in the bases β_{V1}, β_{W1} means that $\forall\, x \in \mathbb{R}^n$

$$T(\beta_{V1}(x)) = \beta_{W1}(M_1 x)$$

$$\begin{tikzcd} V \arrow[r, "T"] & W \\ x \in \mathbb{R}^n \arrow[u, "\beta_{V1}"] \arrow[r, "M_1"'] & y \in \mathbb{R}^m \arrow[u, "\beta_{W1}"'] \end{tikzcd}$$

with $x = \beta_{V1}^{-1}(v)$ this says

$$T(v) = \beta_{W1}(M_1(\beta_{V1}^{-1}(v)))$$

If M_2 represents T in bases β_{V2}, β_{W2} then

$$\forall v \quad T(v) = \beta_{W2}(M_2(\beta_{V2}^{-1}(v)))$$

$$\forall v \quad \beta_{W2}(M_2(\beta_{V2}^{-1}(v))) = \beta_{W1}(M_1(\beta_{V1}^{-1}(v)))$$

(set $x = \beta_{V2}^{-1}(v)$)

$B_V = \beta_{V2}^{-1}\beta_{V1}$

$B_W = \beta_{W2}^{-1}\beta_{W1}$

$$\forall x \qquad \beta_{W2}(M_2(x)) = \beta_{W1}(M_1(\beta_{V1}^{-1}\beta_{V2}(x)))$$

$$M_2(x) = \beta_{W2}^{-1}\beta_{W1} M_1 \beta_{V1}^{-1}\beta_{V2}\, x$$

$$= (B_W M_1 B_V^{-1})\, x$$

$$\boxed{M_2 = B_W M_1 B_V^{-1}}$$