

14 Feb 2022

Change of basis formulae, starting SVD

Recall course enrollment example:

$M =$		course	
	# ugrad	50	
	# grad	20	

$M' =$		course	
	total # stu	70	
	# grad	20	

Change of basis matrix $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. $M' = BM$

But if a linear function has coefficient vector $[2 \ 1]$ in first basis, its coefficient vector in the second one is $[2 \ -1]$.

Def. A based vector space is V with isomorphism $\mathbb{R}^n \xrightarrow{\beta} V$.

Choosing a based vector space on V lets us write elements of V as n -tuples of numbers
 $(v) \quad \beta^{-1}(v)$.

Example. $V =$ vector space of course enrollments.

$\beta_1 : \mathbb{R}^2 \rightarrow V$ basis choice for table M .

$\beta_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) =$ a vector of course enrlls that would be rep'd in M by column vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$=$ a course with 1 ugrad, 0 grad
 $\triangleq \vec{u}$.

$\beta_1 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \triangleq \vec{g} =$ course with 1 grad, 0 ugrad

$\beta_2: \mathbb{R}^2 \rightarrow V$ basis choice for table M'

$\beta_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$ enrollment represented in M'
by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ i.e. 1 total student,
0 grads
 $= \vec{u}$

$\beta_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} =$ enroll rep'd in M' by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ i.e.
0 total students, 1 of whom is grad.
 $= \vec{g} - \vec{u}$

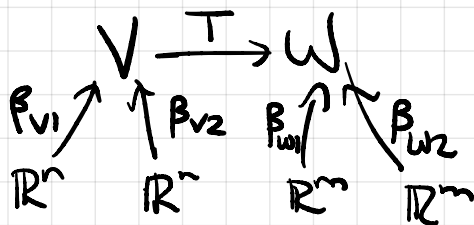
Def. Change of basis matrix from β_1 to β_2 is
the matrix representing the linear transformation

$$\mathbb{R}^n \xrightarrow{\beta_1} V \xrightarrow{\beta_2^{-1}} \mathbb{R}^n \quad \beta_2^{-1} \circ \beta_1$$

$$\beta_2^{-1}(\beta_1(\vec{e}_1)) = \beta_2^{-1}(\vec{u}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \beta_2^{-1}(\beta_1(\vec{e}_2)) &= \beta_2^{-1}(\vec{g}) & B &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \beta_2^{-1}(\vec{u} + \vec{g} - \vec{u}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

What does change-of-basis do to the matrix
representing a linear transformation, $T: V \rightarrow W$?

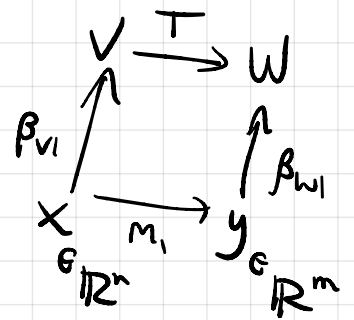


To say that M_1 represents T in the bases β_{V_1}, β_{W_1} means that $\forall x \in \mathbb{R}^n$

$$T(\beta_{V_1}(x)) = \beta_{W_1}(M_1 x)$$

with $x = \beta_{V_1}^{-1}(v)$ this says

$$T(v) = \beta_{W_1}(M_1(\beta_{V_1}^{-1}(v)))$$



If M_2 represents T in bases β_{V_2}, β_{W_2} then

$$\forall v \quad T(v) = \beta_{W_2}(M_2(\beta_{V_2}^{-1}(v)))$$

$$\forall v \quad \beta_{W_2}(M_2(\beta_{V_2}^{-1}(v))) = \beta_{W_1}(M_1(\beta_{V_1}^{-1}(v)))$$

$$(\text{set } x = \beta_{V_2}^{-1}(v))$$

$$\forall x \quad \beta_{W_2}(M_2(x)) = \beta_{W_1}(M_1(\beta_{V_1}^{-1}\beta_{V_2}(x)))$$

$$B_V = \beta_{V_2}^{-1}\beta_{V_1}$$

$$B_W = \beta_{W_2}^{-1}\beta_{W_1}$$

$$M_2(x) = \beta_{W_2}^{-1}\beta_{W_1}M_1\beta_{V_1}^{-1}\beta_{V_2}x$$

$$= (B_W M_1 B_V^{-1})x$$

$$M_2 = B_W M_1 B_V^{-1}$$

