11 Feb 2022 — Matrices and change of basis

Announcements

1. Prof K. extra office hr today
   2:30 — 3:30 Gates 317.
2. Instant Access will charge you for textbook unless you opt out by Mon.

Matrices: two-dimensional arrays of reals

$$M = (M_{ij}) \quad \begin{matrix} 1 \le i \le m & \text{(rows)} \\ 1 \le j \le n & \text{(cols)} \end{matrix}$$

1. Can represent tabular data.

E.g.

	CS1110	...	CS4850	...
# ugrad				
# grad				

2. Can represent linear transformations

$T: \mathbb{R}^n \to \mathbb{R}^m$ represented by M if

$\forall x \quad T(x) =$ the vector $y \in \mathbb{R}^m$ with

$$y_i = \sum_{j=1}^{n} M_{ij} x_j$$

3. Can represent bilinear functions

$A : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$

is bilinear if $A(ax+by, z)$

$= a A(x,z) + b A(y,z)$

and $A(x, ay+bz)$ similarly.

M represents A if

$\forall x \ \forall y \quad A(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij} x_i y_j$

$A(x,y) = \langle x, My \rangle = x^T M y$

$= \langle M^T x, y \rangle =$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$x^T M y = [x_1 \cdots x_m] \begin{bmatrix} & & \\ & M_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$= [x_1 \cdots x_m] \begin{bmatrix} M_{11} y_1 + M_{12} y_2 + \cdots + M_{1n} y_n \\ \vdots \\ M_{m1} y_1 + \cdots + M_{mn} y_n \end{bmatrix}$

$\{$Polynomials of degree $\leq d\}$

$= \{$functions $P(t) = a_0 + a_1 t + \cdots + a_d t^d\}$

form a vector space of dimension $d+1$.

Call this vect space Poly_d.

Here's a bilinear function $\mathrm{Poly}_m \times \mathrm{Poly}_n \to \mathbb{R}$

which is meaningful even if $m \neq n$

$$A(P,Q) = \int_{-1}^{1} P(t)\,Q(t)\,dt.$$

$$A(aP_0 + bP_1, Q) = \int_{-1}^{1} \big(aP_0(t) + bP_1(t)\big)Q(t)\,dt$$

$$= a\int_{-1}^{1} P_0(t)Q(t)\,dt + b\int_{-1}^{1} P_1(t)Q(t)\,dt$$

$$= a\,A(P_0, Q) + b\,A(P_1, Q).$$

Sometimes we want to write a matrix representing "the same thing" as M using a different basis.

Doing this can be confusing because the way to rewrite M depends on what "thing" we're using it to represent.

<u>Example</u>: We can write course enrollments in a table as

	CS 1110	…
# ugrad		
# grad		

or as

	CS 1110	…	CS 4850
total # students			
# grads			

If M denotes a matrix representing the first table of data, then

$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\cdot M$ represents same data

in the 2nd tabular format.

Now suppose the university pays the CS dept $2 for every ugrad we teach
$1 for every grad

If I represent enrollment as $\begin{bmatrix} x \text{ ugrads} \\ y \text{ grads} \end{bmatrix}$

revenue to CS is

$$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

But if I represent it as $\begin{bmatrix} t \text{ tot students} \\ g \text{ grads} \end{bmatrix}$

then revenue to CS is

$$\begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} t \\ g \end{bmatrix}$$

Def. (this class... not a widely used term)

A <u>based</u> <u>vector space</u> is a vect spc V and an isomorphism $\mathbb{R}^n \xrightarrow{\beta} V$ for some $n \in \mathbb{N}$.

Giving vector space V the struct of a based vector space is equiv. to choosing an ordered n-tuple of vectors that form a basis of V.

Two different choices correspond to different bases, and there is a linear function

$$\beta_2^{-1} \circ \beta_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\begin{tikzcd} & V & \\ \mathbb{R}^n \arrow[ur, "\beta_1"] \arrow[rr, bend right, "\beta_2^{-1} \circ \beta_1"'] & & \mathbb{R}^n \arrow[ul, "\beta_2"'] \end{tikzcd}$$

represented by a <u>change of basis</u> matrix, B.

<u>Ex.</u> $V = \{\text{enrollments}\}$.

$\beta_1 : \mathbb{R}^n \longrightarrow V$ $\qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \{\text{ugrad}\}$

$\qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \{\text{grad}\}$.

$\beta_2 : \mathbb{R}^n \longrightarrow V$ $\qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto v_1$

$\qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto v_2$

such that 1 ugrad is represented by $1 \cdot v_1 + 0 \cdot v_2$

and 1 grad is rep'd by $1 \cdot v_1 + 1 \cdot v_2$

$$\therefore \quad ugrad = v_1$$
$$grad = v_1 + v_2$$

$$v_1 = \{ugrad\}$$
$$v_2 = \{grad - ugrad\}.$$