Let x, y be indep random samples from $B^d = \{ x \in \mathbb{R}^d \mid \|x\|_2 \leq 1 \}$.

We seek to ~~calculate~~ bound from above

$$\Pr(\text{angle btw. } x \,\&\, y) \notin \left(\tfrac{\pi}{2} - \varepsilon, \tfrac{\pi}{2} + \varepsilon \right)$$

As you may know, if vectors x, y form angle θ, then

standard inner prod on $\mathbb{R}^d$

$$\langle x, y \rangle = \|x\| \, \|y\| \cos(\theta)$$

$$\theta \notin \left(\tfrac{\pi}{2} - \varepsilon, \tfrac{\pi}{2} + \varepsilon \right) \Longrightarrow |\cos(\theta)| > \sin(\varepsilon) > \varepsilon - \tfrac{1}{3}\varepsilon^3.$$

Assume WLOG y is $\vec{e_1}$. $\langle x, y \rangle = x_1$.

$$\Pr(\text{angle btw. } x \,\&\, y) \notin \left(\tfrac{\pi}{2} - \varepsilon, \tfrac{\pi}{2} + \varepsilon \right)$$

$$\leq \Pr\left(|x_1| > \|x\| \left(\varepsilon - \tfrac{1}{3}\varepsilon^3 \right) \right).$$

Two ways the event $|x_1| > \|x\| \left(\varepsilon - \tfrac{1}{3}\varepsilon^3 \right)$ could happen.

(1) $\|x\| < 1 - \varepsilon \leftarrow$ Probability $< e^{-\varepsilon d}$

(2) $|x_1| > (1-\varepsilon)(\varepsilon - \frac{1}{3}\varepsilon^3)$

$$= \varepsilon - \varepsilon^2 - \frac{1}{3}\varepsilon^3 + \frac{1}{3}\varepsilon^4$$

$$> \frac{1}{2}\varepsilon \quad \text{for small } \varepsilon,$$

e.g. $\varepsilon < \frac{1}{3}$.

Write $\frac{1}{2}\varepsilon$ as $\sqrt{\frac{c}{d}}$ i.e. $c = \frac{\varepsilon^2 d}{4}$

and then $\Pr\left(|x_1| > \sqrt{\frac{c}{d}}\right) < \sqrt{\frac{e}{c}}\, e^{-c/2}$

$$= \sqrt{\frac{4e}{d}} \cdot \varepsilon \cdot e^{-\frac{1}{8}\varepsilon^2 d}$$

We've reached the point of knowing

$$\Pr\left(\text{angle } xy \notin \left(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\right)\right)$$

$$< e^{-\varepsilon d} + \sqrt{\frac{4e}{d}}\, \varepsilon\, e^{-\frac{1}{8}\varepsilon^2 d}$$

$$< 2e^{-\frac{1}{8}\varepsilon^2 d}.$$

Suppose we sample vectors $x_1, x_2, \ldots, x_m \in B^d$ all indep, uniformly random.

What is expected # of pairs x_i, x_j that form angle not in $\left(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\right)$?

pairs

upper bd on small angle prob for 1 pair

Answer: exp. value $\leq \frac{m(m-1)}{2} \, 2 \cdot e^{-\frac{1}{8}\varepsilon^2 d}$

If exp. val $\ll 1$, then with high prob no 2 vectors among $\{x_1, \ldots, x_m\}$ form an angle that differs from $\frac{\pi}{2}$ by $> \varepsilon$.

i.e., if $m^2 \ll e^{\frac{1}{8}\varepsilon^2 d}$

i.e., $m \ll e^{\frac{1}{16}\varepsilon^2 d}$

then this happens with high probability.

Matrices