$\boxed{\text{7 Feb 2022}}$ More on high-dimensional volumes

Public service announcement

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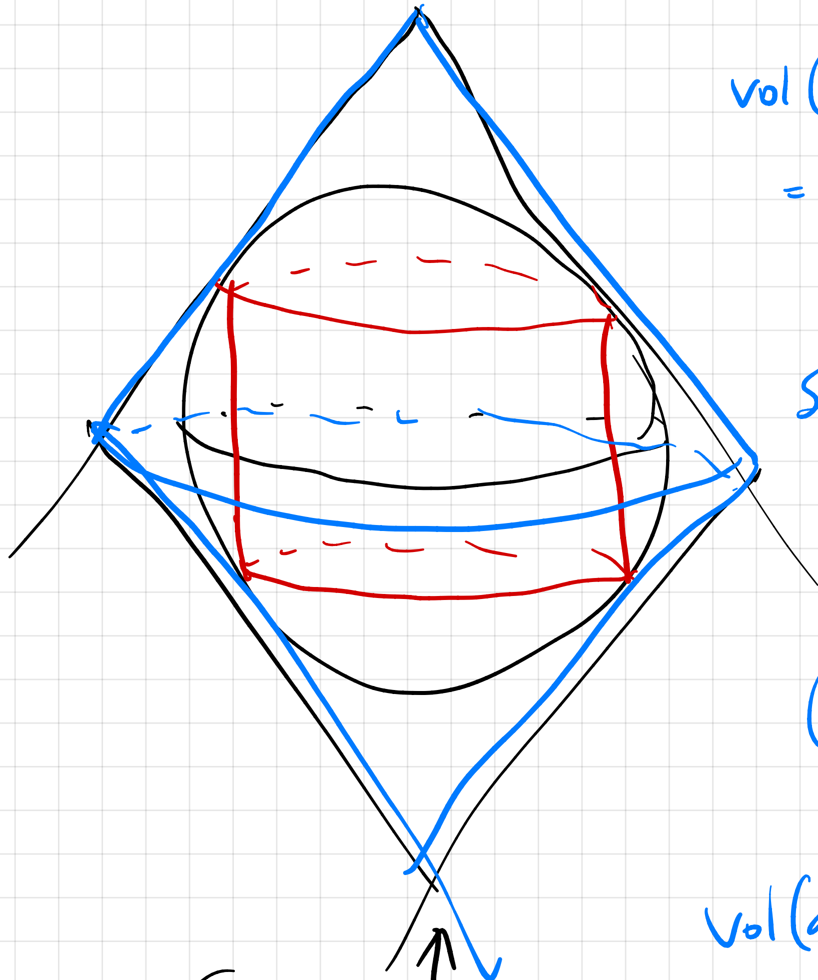
Reminder: if B^d denotes d-dimensional
Euclidean ball of radius 1

$$\mathrm{vol}_d(B^d) = O(d)^{-d/2}$$

Lemma 2.6 in BHK has exact volume formula.

$$\left(\frac{2}{\sqrt{d}}\right)^d < \mathrm{vol}_d(B^d) < \left(\frac{2e}{\sqrt{d}}\right)^d$$



$$\mathrm{vol}(\text{double cone})$$

$$= 2 \cdot \frac{1}{d} \cdot \frac{1}{\varepsilon} \cdot (1-\varepsilon^2)^{-\left(\frac{d-1}{2}\right)} \cdot \mathrm{Vol}(B^{d-1})$$

Set $\varepsilon = \frac{1}{\sqrt{d}}$

$$1-\varepsilon^2 = 1 - \frac{1}{d} = \left(1 + \frac{1}{d-1}\right)^{-1}$$

$$> e^{-\frac{1}{d-1}}.$$

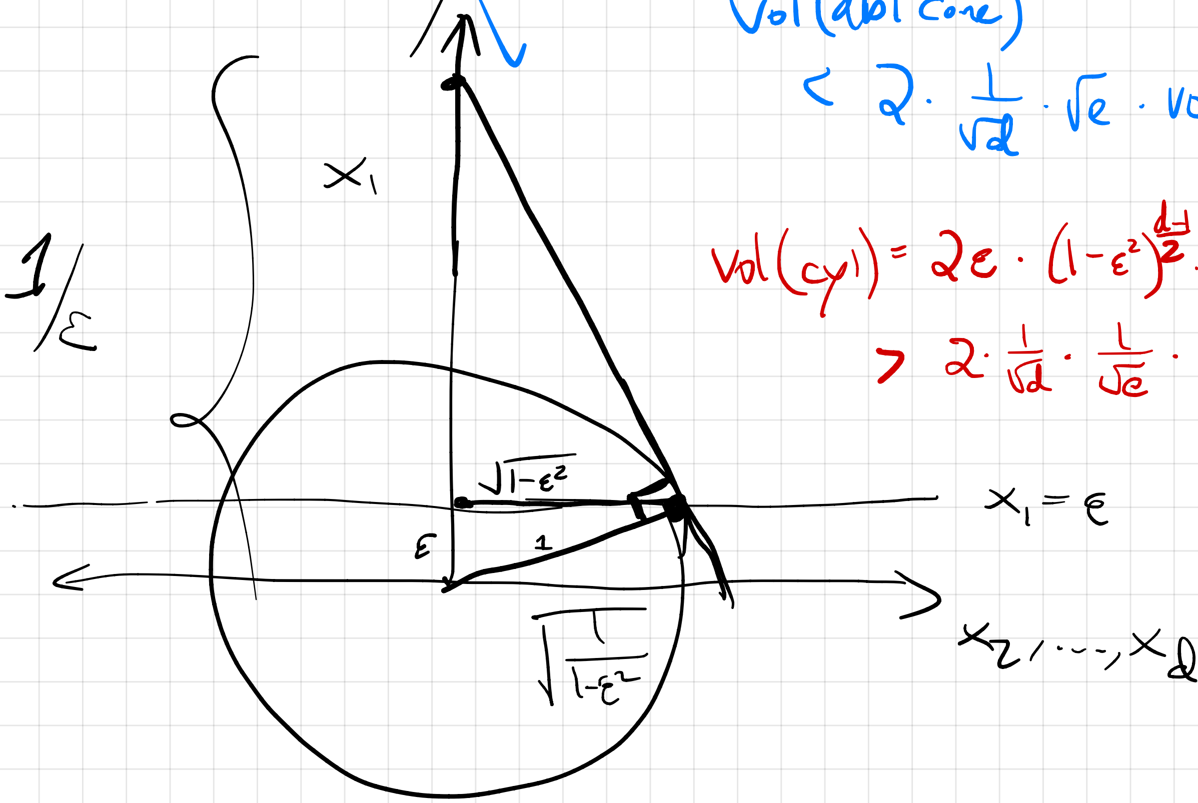
$$(1-\varepsilon^2)^{-\frac{d-1}{2}} < e^{+\frac{1}{\cancel{d-1}} \cdot \frac{\cancel{d-1}}{2}}$$

$$= \sqrt{e}$$

$$\mathrm{vol}(\text{dbl cone}) < 2 \cdot \frac{1}{\sqrt{d}} \cdot \sqrt{e} \cdot \mathrm{vol}(B^{d-1})$$

$$\mathrm{Vol}(\text{cyl}) = 2\varepsilon \cdot (1-\varepsilon^2)^{\frac{d-1}{2}} \cdot \mathrm{Vol}(B^{d-1})$$

$$> 2 \cdot \frac{1}{\sqrt{d}} \cdot \frac{1}{\sqrt{e}} \cdot \mathrm{Vol}(B^{d-1})$$



<u>Lem</u> (Volume Ratio)

$$\frac{2}{\sqrt{e}} \frac{\mathrm{vol}(B^{d-1})}{\sqrt{d}} < \mathrm{vol}(B^d) < 2\sqrt{e} \frac{\mathrm{vol}(B^{d-1})}{\sqrt{d}}.$$

<u>Next.</u> How much of a ball's volume lies in a thin layer near the equator?

constant

$$\text{Thickness of layer} = \sqrt{\frac{c}{d}}$$

dimension

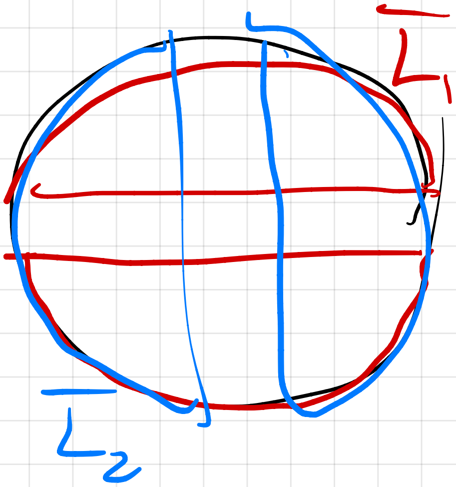
Let $L_i = \left\{ \vec{x} \in B^d \;\middle|\; x_i^2 \leq \frac{c}{d} \right\}.$

The complementary set

$$\overline{L_i} = B \setminus L_i = \left\{ \vec{x} \in B \;\middle|\; x_i^2 > \frac{c}{d} \right\}.$$

How many of the sets $\overline{L_1}, \overline{L_2}, \ldots, \overline{L_d}$ can one element of B belong to?

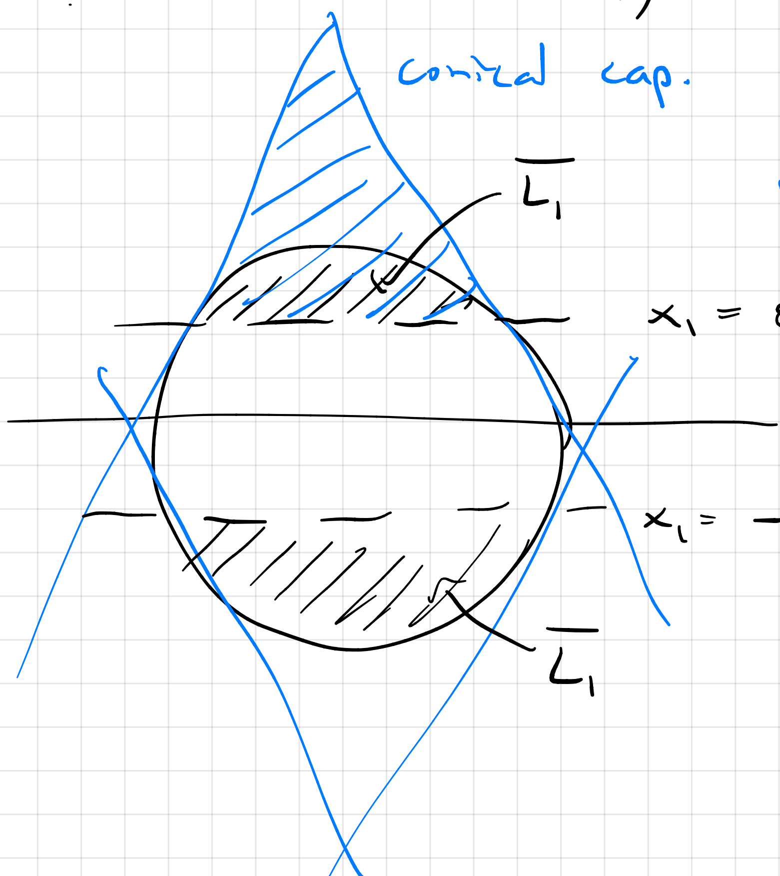
Answer: Less than $\frac{d}{c}$.



$$\sum_{i=1}^{d} \mathrm{vol}(\bar{L}_i) < \frac{d}{c}\cdot \mathrm{vol}(B^d)$$

$$\forall i \quad \mathrm{vol}(\bar{L}_i) < \frac{1}{c}\, \mathrm{vol}(B^d)$$

In fact $\quad \mathrm{vol}(\bar{L}_i) < \sqrt{\frac{e}{c}} \cdot e^{-\frac{c}{2}} \cdot \mathrm{vol}(B^d).$



height $< \frac{1}{\varepsilon} = \sqrt{\frac{d}{c}}$.

radius of base $= \sqrt{1-\varepsilon^2}$

$= \sqrt{1-\frac{c}{d}}$

$< e^{-\frac{c}{2(d-1)}}$ if $c > 4$

$x_1 = \varepsilon = \sqrt{\frac{c}{d}}$

$x_1 = -\varepsilon = -\sqrt{\frac{c}{d}}$

vol(cone cap)

$$= \frac{1}{d} \cdot \mathrm{ht.} \cdot \mathrm{rad}^{d-1} \cdot \mathrm{vol}(B^{d-1})$$

$$< \frac{1}{\sqrt{cd}} \cdot e^{-c/2} \cdot \mathrm{vol}(B^{d-1})$$

$$< \frac{1}{\sqrt{c\cancel{d}}} \cdot e^{-c/2} \cdot \frac{\sqrt{e}}{2} \cdot \cancel{\sqrt{d}} \cdot \mathrm{vol}(B^d)$$

$$= \frac{1}{2}\sqrt{\frac{e}{c}}\, e^{-c/2} \cdot \mathrm{vol}(B^d).$$

$$\boxed{\mathrm{vol}(2 \text{ cones}) < \sqrt{\frac{e}{c}}\, e^{-c/2}\, \mathrm{vol}(B^d)}$$