$\boxed{\text{4 Feb 2022}}$ Geometry of High Dimensional Space

Distance in a vector space is measured by norms.

Def. A norm on vector space V is a function $\|\cdot\|: V \to \mathbb{R}$ satisfying

(i) [non-negativity] $\|x\| \geq 0 \quad \forall x$
with equality only when $x = 0$

(ii) [linear homogeneity] $\|ax\| = |a| \cdot \|x\|$
$\forall x \in V, \quad \forall a \in \mathbb{R}$

(iii) [subadditivity] $\|x+y\| \leq \|x\| + \|y\|$

When these 3 properties are satisfied, then defining $d(x,y) = \|x-y\|$ satisfies symmetry and triangle inequality.

The unit ball of a norm is $B = \{x \mid \|x\| \leq 1\}$.
Check this is a convex set and it is centrally symmetric meaning $x \in B \Leftrightarrow -x \in B$.

Conversely if K is a convex cent. symm set containing 0 in its (topological) interior then there is a norm whose unit ball is K.

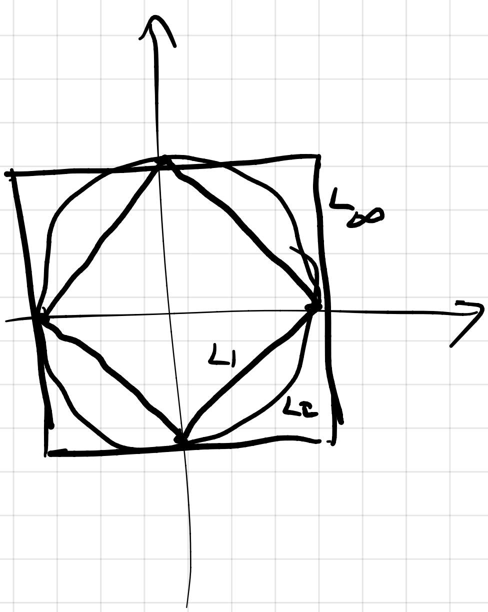
On $\mathbb{R}^n$ there is a family of norms

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} & \text{if } 1 \le p < \infty \\ \max_{1 \le i \le n} |x_i| & \text{if } p = \infty \end{cases}$$

These satisfy: $\|x\|_p$ is a non-increasing function of p as p varies 1 to ∞.

$$B_1 \subset \dots \subset B_p \subset \dots \subset B_\infty$$

unit ball of $\|\cdot\|_p$



$\|\cdot\|_2$ is called Euclidean distance and corresponds to using Pythagorean Theorem to measure distance.

L_p denotes $\mathbb{R}^n$ with the p-norm.

Volumes in d dimensions

Simple facts about volumes.

(1) Additivity; if you dissect a shape into pieces with disjoint interiors, its volume is the sum of their volumes.

* (assuming pieces are "nicely shaped")

(2) Scaling: $\mathrm{vol}_d(\lambda \cdot S) = \lambda^d \cdot \mathrm{vol}_d(S)$

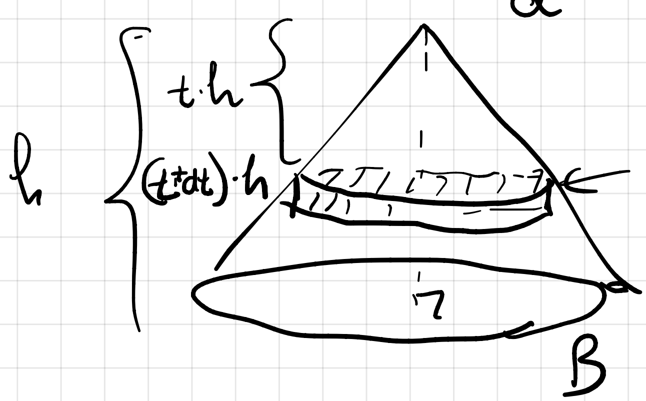
if $S \subset \mathbb{R}^d$ and $\lambda > 0$.

Here

$$\lambda S = \{ \lambda \cdot x \mid x \in S \}.$$

(3) Volume of cone with base B and height h is

$$\frac{1}{d} \cdot h \cdot \mathrm{vol}_{d-1}(B).$$



$\mathrm{Vol}_{d-1}(\text{cross-section}) = t^{d-1}\, \mathrm{vol}_{d-1}(B)$

$$\mathrm{vol}_d(\text{Cone}) = \int_0^1 t^{d-1}\, \mathrm{vol}_{d-1}(B)\, h\, dt$$

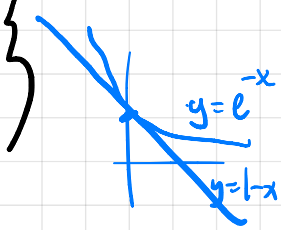
$$= \frac{h}{d}\, \mathrm{vol}_{d-1}(B)$$

Applying these:

<u>Obs 1.</u> Almost all the volume of a d-dimensional Euclidean ball is near the surface!

Let $B = B_2^d(1) = \{ x \in \mathbb{R}^d \mid \|x\|_2 \leq 1 \}$.

Let $S = \{ x \in B \mid \|x\|_2 \geq 1 - \frac{c}{d} \}$



for some $c > 0$.

$1 - x \leq e^{-x}$ $\forall x \in \mathbb{R}$

strict ineq. if $x \neq 0$.

Then $\mathrm{vol}(B \setminus S) < e^{-c} \cdot \mathrm{vol}(B)$.

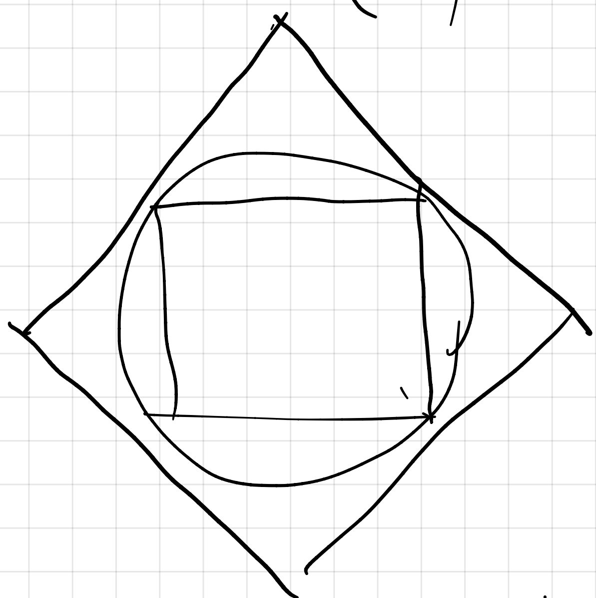
<u>Proof.</u>

$$B \setminus S = B_2^d\left(1 - \frac{c}{d}\right)$$

$$\begin{aligned}
\mathrm{vol}_d(B \setminus S) &= \left(1 - \frac{c}{d}\right)^d \cdot \mathrm{vol}_d(B) \\
&< \left(e^{-c/d}\right)^d \cdot \mathrm{vol}_d(B) \\
&= e^{-c} \cdot \mathrm{vol}_d(B).
\end{aligned}$$

For example when $c = 5$ $e^{-c} < 0.01$, so 99% of the ball's volume is within distance $5/d$ of the boundary.

Obs. 2. $B_\infty^d\left(\frac{1}{\sqrt{d}}\right) \subset B_2^d(1) \subset B_1^d(\sqrt{d})$.



Proof. If $x \in B_\infty^d\left(\frac{1}{\sqrt{d}}\right)$ then $|x_i| \le \frac{1}{\sqrt{d}}$

$$\sum_{i=1}^d x_i^2 \le \sum_{i=1}^d \frac{1}{d} \le 1. \quad \checkmark \quad x \in B_2^d(1)$$

If $x \in B_2^d(1)$, let $y_i = \mathrm{sgn}(x_i)$ so $y_i \in \{\pm 1\}$ and $x_i y_i = |x_i|$.

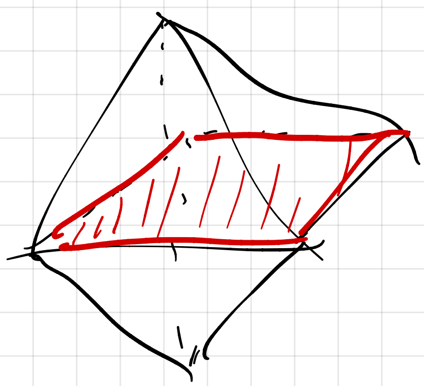
Then
$$\|x\|_1 = \sum_{i=1}^d |x_i| = \sum_{i=1}^d x_i y_i$$
$$\le \left(\sum_{i=1}^d x_i^2\right)^{1/2} \left(\sum_{i=1}^d y_i^2\right)^{1/2}$$
$$\le 1 \cdot \sqrt{d} \quad \checkmark \quad x \in B_1^d(\sqrt{d}).$$

Corollary. $\mathrm{vol}_d\left(B_\infty^d\left(\frac{1}{\sqrt{d}}\right)\right) \le \mathrm{vol}_d\left(B_2^d(1)\right) \le \mathrm{vol}_d\left(B_1^d(\sqrt{d})\right)$

$B_\infty^d\left(\frac{1}{\sqrt{d}}\right)$ is a d-dim'l cube of side length $\frac{2}{\sqrt{d}}$,

$$\mathrm{vol}_d\left(B_\infty\left(\tfrac{1}{\sqrt{d}}\right)\right) = \left(\frac{2}{\sqrt{d}}\right)^d$$

$B_1^d(\sqrt{d})$ has volume $(\sqrt{d})^d \cdot \mathrm{vol}_d(B_1^d(1))$



$B_1^d(1)$ is a union of two congruent cones with height 1 and base $B_1^{d-1}(1)$.

$$\mathrm{vol}_d(B_1^d(1)) = 2 \cdot \frac{1}{d} \cdot \mathrm{vol}_{d-1}(B_1^{d-1}(1))$$

Base case: $\mathrm{vol}_1(B_1^1(1)) = 2.$

$$\mathrm{vol}_d(B_1^d(1)) = 2 \cdot \frac{2}{2} \cdot \frac{2}{3} \cdot \ldots \cdot \frac{2}{d}$$

$$= 2^d / d!$$

Stirling: $d! \sim \sqrt{2\pi d}\left(\frac{d}{e}\right)^d$

Lecture notes: $d! > \sqrt{ed}\left(\frac{d}{e}\right)^d.$

$$\mathrm{vol}_d\left(B_1^d(1)\right) = \frac{2^d}{d!} < 2^d \cdot \frac{1}{\sqrt{ed}} \cdot \left(\frac{e}{d}\right)^d$$

$$< \left(\frac{2e}{d}\right)^d$$

$$\mathrm{vol}_d\left(B_1^d(\sqrt{d})\right) < (\sqrt{d})^d \cdot \left(\frac{2e}{d}\right)^d = \left(\frac{2e}{\sqrt{d}}\right)^d.$$

Now we've got

$$\left(\frac{2}{\sqrt{d}}\right)^d < \mathrm{vol}_d\left(B_2^d(1)\right) < \left(\frac{2e}{\sqrt{d}}\right)^d.$$