28 Jan 2020

Inner products, duals, geometry

Announcements

1. Problem Set 1 due Weds next week.
   ... but you use 2 free slip days to Friday.

2. Office hrs.

   a. TA office hrs start Mon.
      Calendar of office hrs
      will be posted on course website
      tonight or tomorrow.

   b. Prof. Kleinberg extra office hr.
      today, 2:30 - 3:30.
      (Listed on course website.)

3. If you filled out hwk partner survey
   you should have gotten email
   from Abhay Singh. (as2626)

4. You will be allowed to drop lowest
   homework score and quiz score.

Inner products on vector spaces

Def. An inner product on V is a function $V \times V \to \mathbb{R}$ notated by $\langle x, y \rangle$. This must satisfy:

a. Symmetry $\langle x, y \rangle = \langle y, x \rangle$

b. bilinearity $\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle$.

$\langle ax + by, z \rangle$ (similar)

An inner prod is <u>non-degenerate</u> if $\forall x \neq 0 \; \exists y$ s.t. $\langle x, y \rangle \neq 0$.

It is <u>positive-semidefinite</u> (PSD) if $\langle x, x \rangle \geq 0 \; \forall x$.

It is <u>positive definite</u> if PSD and $x = 0$ is the only solution to $\langle x, x \rangle = 0$.

<u>Notes</u>. ① Pos def $\Longrightarrow$ non-degenerate.

② Standard dot product on $\mathbb{R}^n$ satisfies all these properties.

$$\langle x, x \rangle = \sum_{i=1}^{n} x_i^2 \geq 0!$$

③ The Lorentzian inner product $\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2 + \dots + x_n y_n$ is important in relativity theory.

If $\langle \cdot , \cdot \rangle$ is an inner product on V and $x \in V$ then there's a linear function $V \xrightarrow{f_x} \mathbb{R}$ defined by

$$f_x(y) = \langle x, y \rangle.$$

The set of all linear functions $V \to \mathbb{R}$ form a vector space (under adding and scaling functions pointwise) and this v. spc. is denoted by V^* and called the dual of V.

Ex. If $V = \mathbb{R}^n$ then every linear function $V \to \mathbb{R}$ can be written (uniquely) as

$$f(x) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

We'll call the row vector $[a_1 \cdots a_n]$ the coefficient vector of f.

$(\mathbb{R}^n)^* \cong \mathbb{R}^n$

$f \leftrightarrow$ coefficient vector.

Note

$$[a_1 \cdots a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = a_1 x_1 + \cdots + a_n x_n$$
$$= f(\vec{x})$$

Ex. Recall $Z = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x+y+z=0 \right\} \subset \mathbb{R}^3.$

$Z^* = \left\{ \begin{matrix} \text{linear fun} \\ Z \to \mathbb{R} \end{matrix} \right\}$

An example of a function in Z^* is $f(x,y,z) = x+y.$

So the coefficient vector of f is $[1 \; 1 \; 0].$

Another expression for f is $f(x,y,z) = -z.$

So another coefficient vector of f is $[0 \; 0 \; -1].$

Z^* is not a <u>subspace</u> of $(\mathbb{R}^3)^*$,

it is a <u>quotient space</u>

(a vect spc whose elements are equivalence classes of vectors in the bigger space).

Lem. Let V be a fin dim'l vect spc, and $\langle \cdot, \cdot \rangle$ a non-degenerate inner prod. Then $V \cong V^*$ via the bijection

$$\begin{matrix} V & \cong & V^* \\ \Downarrow & & \Downarrow \\ x & \longmapsto & f_x := (y \mapsto \langle x, y \rangle). \end{matrix}$$

Proof. See lecture notes.

Geometry