26 Jan 2022 Vector Spaces

Def. A vector space (over $\mathbb{R}$) is a nonempty set with two operations

- vector addition $V \times V \longrightarrow V$
  $(x, y) \longmapsto x + y$

- scalar multiplication $\mathbb{R} \times V \to V$
  $(a, x) \longmapsto ax$

These satisfy:

(i) Associative

$$(x+y)+z = x+(y+z)$$

$$a(bx) = (ab)x$$

(ii) Commutative

$$x+y = y+x$$

(iii) Distributive

$$(a+b)x = ax+bx$$

$$a(x+y) = ax+ay$$

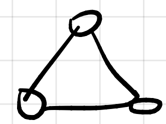
(iv) Identity $1 \cdot x = x$.

The axioms imply that $\forall x$ the vector $\vec{0} \triangleq 0 \cdot x$ is an identity for addition.

Ex. For any set S, $\mathbb{R}^S$ denotes the set of real-valued functions on S.
This is a vector spc under pointwise addition & mult.

E.g. $(f+g)(s) = f(s) + g(s)$

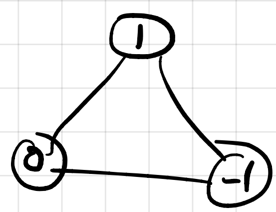
$(a \cdot f)(s) = a \cdot f(s)$.

Ex. Consider the graph $G =$ 

$V(G)$ is a 3-element set.

$\mathbb{R}^{V(G)}$ is the vector space of real-valued labelings of $V(G)$.

E.g.

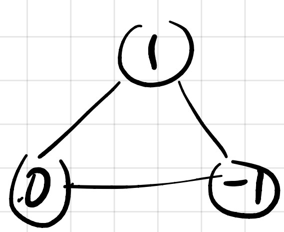


could be represented by $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ in $\mathbb{R}^3$.

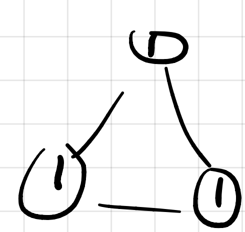
A different representation could be $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Ex. Consider $Z \subset \mathbb{R}^{V(G)}$ consisting of functions on $V(G)$ that sum to zero.

E.g.



$\in Z$



$\notin Z$

The 3-fold symmetry of Z is most apparent when one notates elements of Z as 3-tuples, not 2-tuples.

An <u>isomorphism</u> of vector spaces V, W is a bijection between the elements of V and W, $V \xrightarrow{T} W$, that respects the vector space operations.

$$T(x+y) = T(x) + T(y)$$
$$T(ax) = a\,T(x).$$

E.g. The 6 orderings of $V(G)$ yield 6 different isomorphisms $\mathbb{R}^{V(G)} \longrightarrow \mathbb{R}^3$.

A vector space is <u>finite dimensional</u> iff it is isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$.
(BTW, $\mathbb{R}^0$ is a one-element set $\{\vec{0}\}$.)

<u>Fact:</u> If $V \cong \mathbb{R}^n$ (isomorphic) then $V \not\cong \mathbb{R}^m$ for any $m \neq n$.
The unique n such that $V \cong \mathbb{R}^n$ is called the dimension of V.

(See lecture notes for proof of uniqueness.)

Def. If V is a vect space and $S \subseteq V$ a linear combination of elements of S is a finite sum of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_m x_m$$

where $a_i \in \mathbb{R}$, $x_i \in S$ for $i = 1, \ldots, m$.

The linear comb. is trivial if $a_i = 0 \ \forall i$ otherwise non-trivial.

S is linearly independent if all the non-trivial lin combinations of its elements are $\neq 0$.

A basis of V is a maximal linearly independent set.

The dimension of V is the cardinality of any basis.

(Proving all bases have same cardinality takes work)

If V is a vector space and B is a basis, then the function $T: \mathbb{R}^B \to V$ defined by $T(f) = \sum_{b \in B} f(b) \cdot b$ is an isomorphism.