A data stream consists of elements $a_1, a_2, \ldots, a_n$, where $n$ is the length of the string, and each element $a_i$ is from the alphabet $\{1, 2, \ldots, m\}$.

**Calculate number of distinct elements:**
Lower bound on memory is $m$
Sequence $s, \ldots, a_n$ → state

Every possible subset of symbols corresponds to a different state.
$2^m$ subsets, $2^m$ states, $m = \log_2 m$ memory

$S \subseteq \{1, 2, \ldots, m\}$

$$\begin{array}{cccc}
   \min(S) & 1 & 2 & 3 \\
   \downarrow & \downarrow & \downarrow & \downarrow \\
   m
\end{array}$$

Partition 1 to $m$ by elements of $S$, number of blocks equals $|S| + 1$.

$$\min. \text{ element } = \frac{m}{|S| + 1}, |S| + 1 = \frac{m}{\min}, |S| = \frac{m}{\min} - 1$$

Condition: elements of $S$ are uniformly selected from $\{1, 2, \ldots, m\}$ use hash function $h: \{1, 2, \ldots, m\} \rightarrow \{1, 2, \ldots, m\}$

**Concept of 2-universal:**
A set of hash functions $h$ is 2-universal if for $x, y x \neq y$ and for $u, v$ the probability ($h(x) = u \& h(y) = v$) is independent of $x, y, u$ and $v$.

For each $a, b$ in $[0, m - 1]$ let $h_{a,b}^{(x)} = ax + b \mod M$

**Proof of 2-universal property:**
$h(x) = u$
$h(y) = v$

$$\begin{pmatrix} x & 1 \\ y & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

If $x \neq y$ matrix is non singular and unique solution for $a$ and $b$. 
\[
\text{prob} = \frac{1}{M^2} \text{ independent of } x, y, u \text{ and } v, x \neq y.
\]

Let \( b_1, b_2, \ldots, b_d \) be distinct elements that appear in sequence. Then \( S = \{ h(b_1), h(b_2), \ldots, h(b_d) \} \) is a set of \( d \) random 2-way independent values from \( \{0, 1, 2, \ldots, M - 1\} \).

**Lemma:** With probability of at least \( \frac{2}{3}, \frac{d}{6} \leq \frac{M}{\min} \leq 6d \), where \( d \) is the number of elements.

<table>
<thead>
<tr>
<th>( \leq 1/6 )</th>
<th>( \geq 2/3 )</th>
<th>( \leq 1/6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d/6 )</td>
<td>( 6d )</td>
<td></td>
</tr>
</tbody>
</table>

**Part 1 of proof:**
\[
\text{prob}(\frac{M}{\min} \geq 6d) = \text{prob}(\min \leq \frac{M}{6d}) = \text{prob}(\exists b_i h(b_i) \leq \frac{M}{6d}) = \text{prob}(x \geq 6E(x)) \leq \frac{1}{6}
\]
\[
x_i = 1 \text{ if } h(b_i) \leq \frac{M}{6d}, \quad x_i = 0 \text{ otherwise}
\]
\[
x = \sum_{i=1}^{d} x_i
\]

**Markov inequality:**
\[
E(x_i) = \frac{1}{6d}, \quad E(x) = \frac{1}{6}
\]

**Part 2 of proof:**
\[
\text{prob}(\frac{M}{\min} \leq \frac{d}{6}) = \text{prob}(\min \geq \frac{6M}{d}) = \text{prob}(\forall b_i h(b_i) \geq \frac{6M}{d}) = \text{prob}(y = 0) \leq \text{prob}[|y - E(y)| \geq 6E(y)]
\]
\[
\leq \frac{\text{var}(y)}{E^2(y)} = \frac{1}{E(y)} = \frac{1}{6}
\]
\[
a \sigma = E(y), \quad a = \frac{E(y)}{\sqrt{\text{var}(y)}}
\]
\[
y_i = 0 \text{ if } h(b_i) > \frac{6M}{d}, \quad y_i = 1 \text{ otherwise}
\]
\[
y = \sum_{i=1}^{d} y_i
\]

**Chebyshev’s inequality:**
\[
\text{prob}[|y - E(y)| \geq a \sigma] \leq \frac{1}{a^2}, \quad E(y_i) = \frac{6}{d}, \quad E(y) = 6
\]
\[
\frac{1}{\# \text{ of buckets}}, \quad \# \text{ of buckets} = \frac{d}{6}
\]
\[
\text{var}(y) = \text{dvar}(y_i) \leq dE(y_i^2) = dE(y_i) = E(y)
\]