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**Poisson Distribution:**
\[ P_{\text{Prob}}(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \lambda = \text{avg} \]

**Binomial:**
\[ P_{\text{Prob}}(k) = \binom{n}{k} p^k (1-p)^{n-k} \]

For small \( k \), use \( n-k \sim n \) (where \( \sim \) means about equal)

\[
\binom{n}{k} \approx \frac{n^k}{k!} \left(1 - \frac{d}{n}\right)^{n-k} = \frac{n^k d^k}{k! n^k} \left(1 - \frac{d}{n}\right)^n = \frac{d^k}{k!} e^{-d}
\]

When \( k \) is small, can approximate binomial with Poisson:

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**Back to Last Time...**

1) Mark all vertices as undiscovered & unexplored

2) Pick vertex uniformly at random, mark as discovered

3) Found vertices adjacent to it:

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![Diagram showing process of selecting a vertex from the frontier, adding it to the discovered set, marking it as explored, and moving the frontier out.](image-url)
4) Select vertex from frontier

- Add all adjacent vertices to discovered set
- Mark vertex as explored, move frontier out

Let \( z_i \) be the \# of vertices discovered in the first \( i \) steps of the search

**Distribution of \( z_i \) is binomial \((n-1, 1-(d/n))^i\):**

\( 1-d/n \) = probability not discover it in a given step

\( (1-d/n)^i \) = probability not discover in \( i \) steps

\( 1-(1-d/n)^i \) = probability discovered vertex in \( i \) steps

\[
1 - \left(1 - \frac{1}{n}\right)^i = 1 - \left(1 - \frac{d}{n}\right)^i = 1 - \left(1 + \frac{-1}{n/d}\right)^{\frac{d}{n}}
\]

Using \((1+1/n)^n = e...\)

\[
= 1 - e^{-d/n^i}
\]
So

\[ E(z) = \mathcal{R} \left( 1 - e^{-dN x^2} \right) \]

\[ E(\text{size of frontier}) = \mu - \mu e^{-dN} \]

Normalized:

\[ 1 - e^{-dN} \frac{x}{\mu} \]

If we let \( x = \frac{i}{n} \),

\[ f(x) = 1 - e^{-dN} - x \text{ should match the graph above} \]

\[ f'(x) = dN e^{-dN} - 1 \]

\[ f''(x) = -d^2N e^{-dN} \text{ always negative = curve is convex} \]

\( f(0) = 0 \)

\( f(1) = -e^{-dN} \) (assume \( d > 1 \))

For expected size of frontier:

1. Small components won’t be > \( \frac{\log n}{n} \) (normalized)

2. \( \theta \) is approximately binomial, with Gaussian: \( \theta \mathcal{N} \frac{\mu - \mu e^{-dN}}{\mu} \)

How much does it deviate?

For small \( i \): expected size of frontier grows as \( (d-1)i \)

- In this range binomial distribution can be approximated by Poisson:

\[ p(k) = e^{-\mu_i} \left( \frac{\mu_i^k}{k!} \right) \]

Region: \( [\theta - O\left( \frac{\sqrt{n}}{n} \right), \theta + O\left( \frac{\sqrt{n}}{n} \right)] \)
Questions:

1. How do you know there aren’t two giant components?
2. Does there exist a giant component?

Could there be 2 giant components? NO.

With two growing giant components:

\[
\frac{n}{4} \quad \frac{n}{4} \\
\frac{n}{8} \quad \frac{n}{8}
\]

Some constant probability that the new edge will connect the 2

Non-Uniform Degree Models

Example: cities as vertices, flights between as edges

When do they have a giant component?

Let \( \lambda_i \) be the fraction of vertices of degree \( i \), giant component if:

\[
\sum_{i=1}^{w} i(i - 2)\lambda_i > 0
\]

Why \( (i - 2) \) \( \Rightarrow \) net gain in size of frontier

Degree 1: frontier shrinks

Degree 2: neutral

Degree 4: net gain of 2

Important:
½ vertices degree 1 $\Rightarrow$ 1/3
½ vertices degree 2 $\Rightarrow$ 2/3