Homework assignment 6 is due Friday March 12

1. (a) What is the set of possible harmonic functions on a graph if there are only interior vertices and no external vertices that supply the boundary condition.

(b) Let \( q_x \) be the stationary probability of vertex \( x \) in a random walk on an undirected graph and let \( d_x \) be the degree of vertex \( x \). Show that \( \frac{q_x}{d_x} \) is a harmonic function.

(c) If there are multiple harmonic functions when there are no boundary conditions why is the stationary probability of a random walk on an undirected graph unique?

(d) What is the stationary probability of a random walk on an undirected graph.

2. Given a graph consisting of a single path of five vertices numbered 1 to 5, what is the probability of reaching vertex 1 before vertex 5 when starting at vertex 4.

3. (Optional) Prove that reducing the value of a resistor in a network cannot increase the effective resistance. Try to prove it first for series-parallel networks.

4. Prove that the escape probability \( p_{\text{escape}} = \frac{c_{\text{eff}}}{c_a} \) must be less than one.

5. What is the hitting time \( h_{uv} \) for two adjacent vertices on a cycle of length \( n \)? What is the hitting time if the edge \( u,v \) is removed?

(Some hints on solving non-homogeneous second order difference equations: If you want to solve a difference equation of the form:
\[ c_1 h_{(i-1)} + c_2 h_i + c_3 h_{(i+1)} = d \]
then the solution is the superposition of the solution to the homogeneous equation plus a particular solution to the non-homogeneous.
\[ h_i = h_i^{\text{hom}} + h_i^{\text{part}} \]
Where \( h_i^{\text{hom}} \) is the solution to \( c_1 h_{(i-1)} + c_2 h_i + c_3 h_{(i+1)} = 0 \) and \( h_i^{\text{part}} \) is any function of \( i \) that satisfies \( c_1 h_{(i-1)} + c_2 h_i + c_3 h_{(i+1)} = d \) (Hint: for the difference equation that might arise in your problem try using \( h_i^{\text{part}} = i^2 \)). The way of solving homogeneous difference equations was described in class and can also be found in last years notes:

After you find a general solution \( h_i \) you have to calculate the constants that remain in your solution by taking border cases of your recursion, e.g. what happens at \( h_1 \) or \( h_0 \) or \( h_n \) etc.)

6. Consider the set of integers \( \{1, 2, \cdots, n\} \). How many draws \( d \) with replacement are necessary so that every integer is drawn?
This figure is for the next homework: