Correlation in tree
The (ferromagnetic) Ising model

The Ising model is a graphical model or pair wise random Markov field consisting of an undirected graph with variables associated with the vertices. The graph, usually a lattice, has a value of ±1 called spin assigned to each variable at a vertex. The probability (Gibbs measure) of a given configuration of spins is proportional to $e^{\beta \sum_{edges(i,j)} x_i x_j} = \prod_{(i,j)\in E} e^{\beta x_i x_j}$ where $x_i = \pm 1$ is the value associated with vertex i. Thus

$$p(x_1, x_2, \cdots, x_n) = \tfrac{1}{Z} \prod_{(i,j)\in E} e^{\beta x_i x_j} = \tfrac{1}{Z} e^{\beta \sum_{(i,j)\in E} x_i x_j}$$

where Z is a normalization constant.

The value of the summation is simply the difference in the number of edges whose vertices have the same spin minus the number of edges whose vertices have opposite spin. The constant β is viewed as inverse temperature. High temperature corresponds to a low value of β. At low temperature, high β, adjacent vertices have identical spins where as at high temperature the spins of adjacent vertices are uncorrelated.

One question of interest is given the above probability distribution what is the correlation between two variables say x_i and x_j. To answer this, we want to determine the $\text{Prob}(x_i = 1)$ as a function of $\text{Prob}(x_j = 1)$. If $\text{Prob}(x_i = 1) = \tfrac{1}{2}$ independent of the value of $\text{Prob}(x_j = 1)$, we say the values are uncorrelated.

Consider the special case where the graph G is a tree. In this case a phase transition occurs at $\beta_0 = \tfrac{1}{2} \ln \tfrac{d+1}{d-1}$ where *d* is the degree of the tree. For a sufficiently tall tree and for $\beta > \beta_0$ the probability that the root has value +1 is bounded away from ½ and depends on whether the majority of leaves have value $+1$ or -1. For $\beta < \beta_0$ the probability that the root has value +1 is ½ independent of the values at the leaves of the tree.

Consider a height one tree of degree *d*. If *i* of the leaves have spin +1 and *d-i* have spin -1, then the probability of the root having spin +1 is proportional to

$$e^{i\beta - (d-i)\beta} = e^{(2i-d)\beta}.$$

If the probability of a leaf being +1 is *p*, then the probability of i leaves being +1 and *d-i* being -1 is

$$\binom{d}{i} p^i (1-p)^{d-i}$$

Thus, the probability of the root being +1 is proportional to

$$A=\sum_{i=1}^{d}\binom{d}{i}p^i\left(1-p\right)^{d-i}e^{(2i-d)\beta}=e^{-d\beta}\sum_{i=1}^{d}\binom{d}{i}\left(pe^{2\beta}\right)^i\left(1-p\right)^{d-i}=e^{-d\beta}\left[pe^{2\beta}+1-p\right]^d$$

and the probability of the root being –1 is proportional to

$$B=\sum_{i=1}^{d}\binom{d}{i}p^i\left(1-p\right)^{d-i}e^{-(2i-d)\beta}=e^{-d\beta}\sum_{i=1}^{d}\binom{d}{i}\left(p\right)^i\left[\left(1-p\right)e^{2\beta}\right]^{d-i}=e^{-d\beta}\left[p+\left(1-p\right)e^{2\beta}\right]^d.$$

The probability of the root being +1 is

$$\begin{aligned}q&=\frac{A}{A+B}\\&=\frac{\left[pe^{2\beta}+1-p\right]^d}{\left[pe^{2\beta}+1-p\right]^d+\left[p+\left(1-p\right)e^{2\beta}\right]^d}\\&=\frac{C}{D}\end{aligned}$$

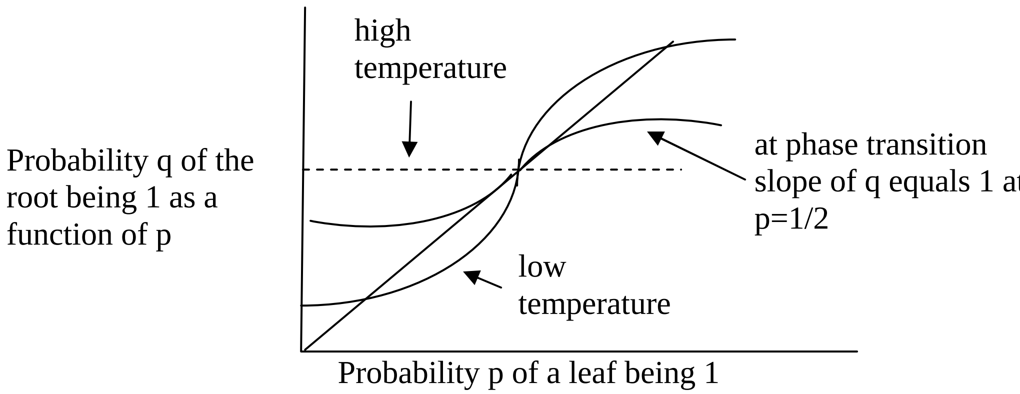
where $C=\left[pe^{2\beta}+1-p\right]^d$ and $D=\left[pe^{2\beta}+1-p\right]^d+\left[p+\left(1-p\right)e^{2\beta}\right]^d$.

Now the slope of the probability of the root being 1 with respect to the probability of a leaf being 1 in this height one tree is

$$\frac{\partial q}{\partial p}=\frac{D\frac{\partial C}{\partial p}-C\frac{\partial D}{\partial p}}{D^2}$$

At high temperature the probability q of the root of this height one tree being 1 is 1/2 independent of p. At low temperature q goes from low probability of 1 below p=1/2 to high probability of 1 above p=1/2.

How consider a very tall tree. If the p is the probability that a root has value +1, we can iterate the formula for the height one tree and we see that at low temperature the probability of the root being one converges to some value. At high temperature the probability of the root being one is ½ independent of p. At the phase transition, the slope of q at p=1/2 is one.



Since the slope of the function q(p) at p=1/2 when the phase transition occurs is one we can solve $\frac{\partial q}{\partial p}=1$ for the value of β when the phase transition occurs. First we show that $\left.\frac{\partial D}{\partial p}\right|_{p=\frac{1}{2}}=0$.

$$D=\left[pe^{2\beta}+1-p\right]^{d}+\left[p+(1-p)e^{2\beta}\right]^{d}$$

$$\tfrac{\partial D}{\partial p}=d\left[pe^{2\beta}+1-p\right]^{d-1}\left(e^{2\beta}-1\right)+d\left[p+(1-p)e^{2\beta}\right]^{d-1}\left(1-e^{2\beta}\right)$$

$$\left.\tfrac{\partial D}{\partial p}\right|_{p=\frac{1}{2}}=\tfrac{d}{2^{d-1}}\left[e^{2\beta}+1\right]^{d-1}\left(e^{2\beta}-1\right)+\tfrac{d}{2^{d-1}}\left[1+e^{2\beta}\right]^{d-1}\left(1-e^{2\beta}\right)=0$$

Then

$$\left.\frac{\partial q}{\partial p}\right|_{p=\frac{1}{2}}=\frac{\partial}{\partial p}\left.\frac{D\frac{\partial C}{\partial p}-C\frac{\partial D}{\partial p}}{D^{2}}\right|_{p=\frac{1}{2}}=\left.\frac{\frac{\partial C}{\partial p}}{D}\right|_{p=\frac{1}{2}}$$

$$=\left.\frac{d\left[pe^{2\beta}+1-p\right]^{d-1}\left(e^{2\beta}-1\right)}{\left[pe^{2\beta}+1-p\right]^{d}+\left[p+(1-p)e^{2\beta}\right]^{d}}\right|_{p=\frac{1}{2}}$$

$$=\frac{d\left[\frac{1}{2}e^{2\beta}+\frac{1}{2}\right]^{d-1}\left(e^{2\beta}-1\right)}{\left[\frac{1}{2}e^{2\beta}+\frac{1}{2}\right]^{d}+\left[\frac{1}{2}+\frac{1}{2}e^{2\beta}\right]^{d}}$$

$$\frac{d\left(e^{2\beta}-1\right)}{1+e^{2\beta}}$$

Setting

$$\frac{d\left(e^{2\beta}-1\right)}{1+e^{2\beta}}=1$$

yields

$$d\left(e^{2\beta}-1\right)=1+e^{2\beta}$$
$$e^{2\beta}=\tfrac{d+1}{d-1}$$
$$\beta=\tfrac{1}{2}\ln\tfrac{d+1}{d-1}$$

Shape of q as a function of p.

To complete the argument we need to show that q is a monotonic function of p. To see this write $q=\frac{1}{1+\frac{B}{A}}$. A is monotonically increasing function of p and B is monotonically decreasing. From this it follows that q is monotonically increasing.

Note: The joint probability distribution for the tree is of the form $e^{\beta\sum\limits_{edges(i,j)} x_i x_j}=\prod\limits_{(i,j)\in E} e^{\beta x_i x_j}$.

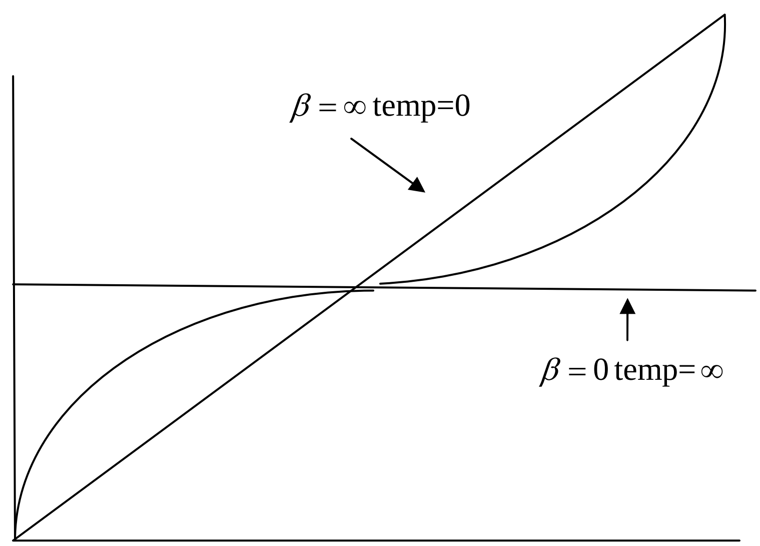
Suppose we now that x_1 has value 1 with probability p. Then we could define a function ϕ , called evidence, such that $\phi(x_1)=\begin{cases} p & \text{for } x_1=1 \\ 1-p & \text{for } x_1=-1 \end{cases}=\left(p-\tfrac{1}{2}\right)x_1+\tfrac{1}{2}$ and multiply the joint probability function by ϕ. Note however that the marginal probability of x_1 is not p. In factor it may be further from p after multiplying the conditional probability function by the function ϕ.

Note that it the iteration going from p to q we do not get the true marginal probabilities at each level since we ignore the effect of the portion of the tree above. However, when we get to the root we do get the true marginal for the root. To get the true marginal's for the interior nodes we need to send messages down from the root.

Exercises

Exercise 1: For a tree with degree one, a chain of vertices, is there a phase transition? Work out mathematical what happens.

Solution: Let the vertices of the chain be $x_1, x_2, \cdots, x_n$. The joint probability distribution is $p(x_1, x_2, \cdots, x_n) = \frac{1}{Z}\prod_{i=1}^{n-1} e^{\beta x_1 x_{1+1}}$. If we multiply the joint probability distribution by the evidence function $\phi(x_n) = \left(p - \frac{1}{2}\right)x_n + \frac{1}{2}$ we get $p(x_1, x_2, \cdots, x_n) = \frac{1}{Z}\prod_{i=1}^{n-1} e^{\beta x_1 x_{1+1}}\phi(x_n)$ and summing over x_n we get $p(x_1, x_2, \cdots, x_{n-1}) = \frac{1}{Z}\prod_{i=1}^{n-2} e^{\beta x_1 x_{1+1}}\sum_{x_n} e^{\beta x_{n-1}x_n}\phi(x_n) = \frac{1}{Z}\prod_{i=1}^{n-2} e^{\beta x_1 x_{1+1}}\left[(1-p)e^{-\beta x_{n-1}} + pe^{\beta x_{n-1}}\right]$.
The function $\left[(1-p)e^{-\beta x_{n-1}} + pe^{\beta x_{n-1}}\right]$ is evidence for x_{n-1}. If we normalize it so it is a probability function we get $\frac{\left[(1-p)e^{-\beta x_{n-1}} + pe^{\beta x_{n-1}}\right]}{e^{-\beta} + e^{\beta}}$. For various values of β we get the figure below.



Exercise 2: Consider an n by n lattice. By experiment determine the value of β at which a phase transition occurs for the correlation of the spin of the origin vertex as a function of the spin on the boundary of the square.

Exercise 3: