8.4 Combining Rankings

A ranking is a complete ordering in the sense that for every pair of items *a* and *b*, either *a* is preferred to *b*, *b* is preferred to *a*, or *a* and *b* are tied. Furthermore a ranking is transitive in that $a>b$, $b>c$ implies $a>c$. Suppose there are n individuals or voters and m items to be ranked. Each voter produces a ranked list of the items. From the set of *n* ranked lists can one construct a single ranking of the *m* items? Assume the method of producing a global ranking is required to satisfy the following three axioms.

Non dictatorship – The algorithm cannot always simply select one individual's ranking.

Unanimity - If every individual prefers *a* to *b*, then the global ranking must prefer *a* to *b*.

Independent of irrelevant alternatives – If individuals modify their rankings but keep the order of *a* and *b* unchanged, then the global order of *a* and *b* should not change.

Arrow showed that no such algorithm exists satisfying the above axioms.

Example: Merging ranked lists is non trivial. Suppose there are three individuals who rank three items a, b, and c.

individual	first item	second item	third item
1	*a*	*b*	*c*
2	*b*	*c*	*a*
3	*c*	*a*	*b*

Suppose our algorithm tried to rank the items by first comparing *a* to *b* and then comparing *b* to *c*. In comparing *a* to *b*, two of the individuals prefer *a* better than *b*. In comparing *b* to *c*, again two individuals prefer *b* to *c*. Now by transitivity one would hope that the individuals would prefer *a* to *c*, but such is not the case. We come to the illogical conclusion that *a* is preferred to *b*, *b* is preferred to *c* and *c* is preferred to *a*.

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Theorem: (**Arrow**) Any algorithm for creating a global ranking of three or more elements that satisfies unanimity and independence of irrelevant alternatives is a dictatorship.

Proof: Let *a*, *b*, and *c* be distinct items. Consider a set of rankings in which each person ranks *b* either first or last. Some may rank *b* first and others may rank *b* last. For this set of rankings the global ranking must put *b* first or last. Suppose to the contrary that *b* is not first or last in the global ranking. Then there exist *a* and *c* where the global ranking puts $a \geq b$ and $b \geq c$. By transitivity, $a \geq c$ in the global ranking. By independence of irrelevant alternatives, the global ranking would continue to rank $a \geq b$ and $b \geq c$ even if all individuals moved *c* above *a* since that would not change the relative order of *a* and *b* or the relative order of *b* and *c*. . But then by unanimity, the global ranking would put $c > a$, a contradiction. We conclude that the global ranking puts *b* first or last.

Consider a set of rankings in which every individual ranks *b* last. By unanimity, the global ranking must also. Let the voters, one by one, move *b* from bottom to top leaving the other rankings in place. By unanimity, *b* must eventually move to the top. Let *v* be the first voter whose change causes the global ranking of *b* to change.

We now argue that *v* is a dictator. First *v* is a dictator for any pair *ac* not involving *b*. We will refer to two rankings. The first is the ranking prior to v moving b from the bottom to the top and the second is the ranking just after v has moved v to the top. Choose any pair *ac* where *a* is above *c* in *v*'s ranking. Let *v* modify his ranking that exists just after moving v to the top by moving *a* above *b* so that $a > b > c$ in *v*'s ranking and let all other voters modify their rankings arbitrarily while leaving *b* in its extreme position. By independence of irrelevant alternatives the global ranking puts $a > b$ since all individual *ab* votes are still the same as just before *v* moved *b* to the top of his ranking. At that time the global ranking placed $a > b$. Similarly b>c in the global ranking since all individual *bc* votes are the same as just after *v* moved *b* to the top. By transitivity the global ranking must put $a > c$ and thus the global ranking of *a* and *c* agrees with *v*. Note that the global ranking of *a* and *c* are independent of where *v* places *b* since placing *b* does not change the relative order of *a* and *c*. Thus we conclude that for all a and c the global ranking agrees with v independent of how the other rankings rearrange their order as long as b remains at its extreme position in these rankings. Note that *v* can change the global relative order of *a* and *b* by moving *b* from bottom to top.

The global ranking will agree with the column v_b as long as the b's are in their appropriate positions. Consider moving the positions of the b's to arbitrary locations. By independence of irrelevant alternatives this does not change the global order of any elements except for b. Thus, column v_b is a dictator for the ordering al all elements except for b. Repeat the above argument interchanging the roles of b and c. Then some column is the dictator for the order of all elements except for c. Note that moving b down the column v_b interchanges the order of a and b. This implies that $v_b = v_c$ and thus column v_b is a dictator.

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Exercise: Prove that the global ranking agrees with column v_b even if b is moved down through the column.

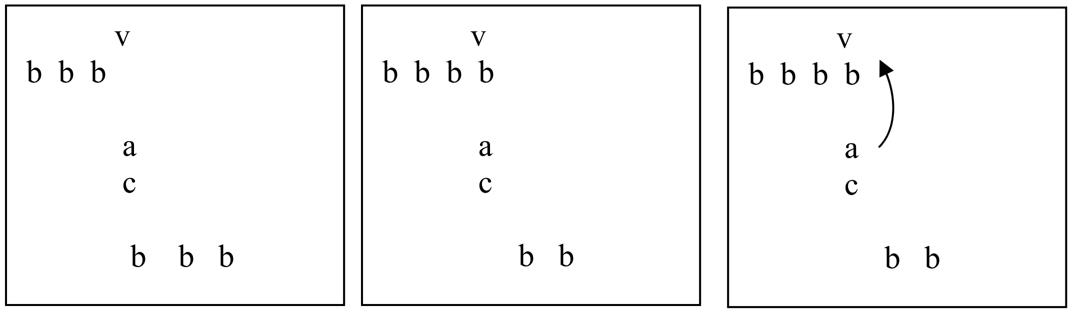

Figure XXX: Illustration that v is a dictator. When v moves b to the top the global order moves b to the top. The global order must place a above c since moving b to the top does not affect the order of a and c. In the left most figure the global order place b<a since the order of b and a are the same as in the left most figure where b is at the bottom in the global order. In the middle figure we see that the global order must place c<b since b is at the top. The global order of a and c do not change in the three pictures. This argument holds independent of how the other voter rearrange a and c as long as they do not move b from its extreme position.

Exercise: Show that the three axioms: non dictator, unanimity, and independence of irrelevant alternatives are independent.

Note that if the ranking's of individual voters are feed to the computer program in a different order, then the dictator will be a different voter. Suppose there were seven voters. The dictator might always be the fourth even if one permuted the order of voters.

Hare system for voting
see http://bcn.boulder.co.us/government/approvalvote/altvote.html

Consider the following situation in which there are 21 voters that fall into four categories. Voters within a category rank individuals in the same order.

category	number of voters in category	preference order
1	7	abcd
2	6	bacd
3	5	cbad
4	3	dcba

The Hare system would first eliminate *d* since *d* gets only three rank 1 votes. Then it would eliminate *b* since *b* gets only 6 rank 1 votes. At this point *a* is declared the winner since *a* has 13 votes to *c*'s 8 votes.

Now assume that category 4 voters who prefer *b* to *a* move *a* up to first place. Then the election proceeds as follows. In round one *d* is eliminated since it gets no rank 1 votes. Then *c* with five votes is eliminated and *b* is declared the winner with 11 votes. Note that by moving *a* up, category 4 voters were able to deny *a* the election and get *b* to win who they preferred over *a*.

Exercise: Does the axiom of independence of irrelevant alternatives make sense? What if there were three rankings of five items. In the first two rankings A is number one and B is number two. In the third ranking B is number one and A is number five. one might compute an average score where a low score is good. A gets a score of 1+1+5=7 and B gets a score of 2+2+1=5 and B is ranked number one in the global raking. Now if the third ranker moves A up to the second position A's score becomes 1+1+2=4 and the global ranking changes. Is there some alternative axiom to replace independence of irrelevant alternatives?

References

Arrow's impossibility theorem – Wikipedia

John Geanakoplos, "Three brief proofs of Arrow's Impossibility Theorem"