8.4 Combining Rankings

A ranking is a complete ordering in the sense that for every pair of items $a$ and $b$, either $a$ is preferred to $b$, $b$ is preferred to $a$, or $a$ and $b$ are tied. Furthermore a ranking is transitive in that $a > b$, $b > c$ implies $a > c$. Suppose there are $n$ individuals or voters and $m$ items to be ranked. Each voter produces a ranked list of the items. From the set of $n$ ranked lists can one construct a single ranking of the $m$ items? Assume the method of producing a global ranking is required to satisfy the following three axioms.

**Non dictatorship** – The algorithm cannot always simply select one individual’s ranking.

**Unanimity** - If every individual prefers $a$ to $b$, then the global ranking must prefer $a$ to $b$.

**Independent of irrelevant alternatives** – If individuals modify their rankings but keep the order of $a$ and $b$ unchanged, then the global order of $a$ and $b$ should not change.

Arrow showed that no such algorithm exists satisfying the above axioms.

**Example**: Merging ranked lists is non trivial. Suppose there are three individuals who rank three items $a$, $b$, and $c$.

<table>
<thead>
<tr>
<th>individual</th>
<th>first item</th>
<th>second item</th>
<th>third item</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
</tr>
<tr>
<td>2</td>
<td>$b$</td>
<td>$c$</td>
<td>$a$</td>
</tr>
<tr>
<td>3</td>
<td>$c$</td>
<td>$a$</td>
<td>$b$</td>
</tr>
</tbody>
</table>

Suppose our algorithm tried to rank the items by first comparing $a$ to $b$ and then comparing $b$ to $c$. In comparing $a$ to $b$, two of the individuals prefer $a$ better than $b$. In comparing $b$ to $c$, again two individuals prefer $b$ to $c$. Now by transitivity one would hope that the individuals would prefer $a$ to $c$, but such is not the case. We come to the illogical conclusion that $a$ is preferred to $b$, $b$ is preferred to $c$ and $c$ is preferred to $a$.

**Theorem**: (Arrow) Any algorithm for creating a global ranking of three or more elements that satisfies unanimity and independence of irrelevant alternatives is a dictatorship.

**Proof**: Let $a$, $b$, and $c$ be distinct items. Consider a set of rankings in which each person ranks $b$ either first or last. Some may rank $b$ first and others may rank $b$ last. For this set of rankings the global ranking must put $b$ first or last. Suppose to the contrary that $b$ is not first or last in the global ranking. Then there exist $a$ and $c$ where the global ranking puts $a \geq b$ and $b \geq c$. By transitivity, $a \geq c$ in the global ranking. By independence of irrelevant alternatives, the global ranking would continue to rank $a \geq b$ and $b \geq c$ even if all individuals moved $c$ above $a$ since that would not change the relative order of $a$ and $b$. ■
or the relative order of $b$ and $c$. But then by unanimity, the global ranking would put $c > a$, a contradiction. We conclude that the global ranking puts $b$ first or last.

Consider a set of rankings in which every individual ranks $b$ last. By unanimity, the global ranking must also. Let the voters, one by one, move $b$ from bottom to top leaving the other rankings in place. By unanimity, $b$ must eventually move to the top. Let $v$ be the first voter whose change causes the global ranking of $b$ to change.

We now argue that $v$ is a dictator. First $v$ is a dictator for any pair $ac$ not involving $b$. We will refer to two rankings. The first is the ranking prior to $v$ moving $b$ from the bottom to the top and the second is the ranking just after $v$ has moved $v$ to the top. Choose any pair $ac$ where $a$ is above $c$ in $v$'s ranking. Let $v$ modify his ranking that exists just after moving $v$ to the top by moving $a$ above $b$ so that $a > b > c$ in $v$'s ranking and let all other voters modify their rankings arbitrarily while leaving $b$ in its extreme position. By independence of irrelevant alternatives the global ranking puts $a > b$ since all individual $ab$ votes are still the same as just before $v$ moved $b$ to the top of his ranking. At that time the global ranking placed $a > b$. Similarly $b > c$ in the global ranking since all individual $bc$ votes are the same as just after $v$ moved $b$ to the top. By transitivity the global ranking must put $a > c$ and thus the global ranking of $a$ and $c$ agrees with $v$. Note that the global ranking of $a$ and $c$ are independent of where $v$ places $b$ since placing $b$ does not change the relative order of $a$ and $c$. Thus we conclude that for all $a$ and $c$ the global ranking agrees with $v$ independent of how the other rankings rearrange their order as long as $b$ remains at its extreme position in these rankings. Note that $v$ can change the global relative order of $a$ and $b$ by moving $b$ from bottom to top.

The global ranking will agree with the column $v_b$ as long as the $b$'s are in their appropriate positions. Consider moving the positions of the $b$'s to arbitrary locations. By independence of irrelevant alternatives this does not change the global order of any elements except for $b$. Thus, column $v_b$ is a dictator for the ordering of all elements except for $b$. Repeat the above argument interchanging the roles of $b$ and $c$. Then some column is the dictator for the order of all elements except for $c$. Note that moving $b$ down the column $v_b$ interchanges the order of $a$ and $b$. This implies that $v_b = v_c$ and thus column $v_b$ is a dictator.

**Exercise**: Prove that the global ranking agrees with column $v_b$, even if $b$ is moved down through the column.
8.4 Combining Rankings

Exercise: Show that the three axioms: non dictator, unanimity, and independence of irrelevant alternatives are independent.

Note that if the ranking’s of individual voters are feed to the computer program in a different order, then the dictator will be a different voter. Suppose there were seven voters. The dictator might always be the fourth even if one permuted the order of voters.

Hare system for voting
see http://bcn.boulder.co.us/government/approvalvote/altvote.html

Consider the following situation in which there are 21 voters that fall into four categories. Voters within a category rank individuals in the same order.

<table>
<thead>
<tr>
<th>category</th>
<th>number of voters in category</th>
<th>preference order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>abcd</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>bacd</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>cbad</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>dcba</td>
</tr>
</tbody>
</table>

The Hare system would first eliminate $d$ since $d$ gets only three rank 1 votes. Then it would eliminate $b$ since $b$ gets only 6 rank 1 votes. At this point $a$ is declared the winner since $a$ has 13 votes to $c$’s 8 votes.
Now assume that category 4 voters who prefer \(b\) to \(a\) move \(a\) up to first place. Then the election proceeds as follows. In round one \(d\) is eliminated since it gets no rank 1 votes. Then \(c\) with five votes is eliminated and \(b\) is declared the winner with 11 votes. Note that by moving \(a\) up, category 4 voters were able to deny \(a\) the election and get \(b\) to win who they preferred over \(a\).

**Exercise**: Does the axiom of independence of irrelevant alternatives make sense? What if there were three rankings of five items. In the first two rankings A is number one and B is number two. In the third ranking B is number one and A is number five. one might compute an average score where a low score is good. A gets a score of 1+1+5=7 and B gets a score of 2+2+1=5 and B is ranked number one in the global ranking. Now if the third ranker moves A up to the second position A’s score becomes 1+1+2=4 and the global ranking changes. Is there some alternative axiom to replace independence of irrelevant alternatives?

**References**

Arrow’s impossibility theorem – Wikipedia

John Geanakoplos, “Three brief proofs of Arrow’s Impossibility Theorem”