Homework 7 due March 13

Do the project, problem 1 and select four problems from the list below that interest you and solve them.

Project: Do a careful simulation to see what happens in the case of a grown graph with edges added at each time unit between two vertices selected uniformly at random with probability δ when δ approaches the critical value of 1/8. Plot the average size of finite components with respect to δ. What difficulty do you run into with your simulation and how have you solved it?

Problems

1. Suppose you have a large undirected graph with billions of vertices and you want to know the distribution of component sizes. For example, what fraction of the vertices is isolated, what fraction is in components of size two, etc. You decide to determine this by sampling. You select a vertex and do a breadth first search starting from the vertex to find the size of the component containing the vertex. You repeat this process for a few thousand vertices. How do you estimate the percentages of components that are of various sizes?

2. For the Fibonacci sequence prove that $f_n = \left\lfloor \frac{\sqrt{5}}{5}\theta_1^n \right\rfloor$ for all $n \geq 1$ where θ_1 is the larger of the two roots of the equation $x^2 + x - 1 = 0$.

3. In the growing graph we developed the equation

$$-x + \sum_{k=11}^{\infty} k a_k x^k + 2\delta x \sum_{k=1}^{\infty} a_k k^2 x^{k-1} = \delta \sum_{k=1}^{\infty} k x^k \sum_{j=1}^{k-1} j\left(k-j\right) a_j a_{k-j}$$

From the above equation derive the equation for $g'\left(x\right)$ where $g\left(x\right) = \sum_{k=1}^{\infty} k a_k x^k$.

4. Consider a static random graph (not a grown graph) with degree distribution

$$p_k = \frac{\left(2\delta\right)^k}{\left(1+2\delta\right)^{k+1}}.$$

Use the Molloy Reed formula $\sum_{i=0}^{\infty} i\left(i-2\right) p_i$ to show that the phase transition where a giant component appears is $\delta = \frac{1}{4}$. You need to show that the formula is negative for $\delta < \frac{1}{4}$ and positive for $\delta > \frac{1}{4}$.

5. For the grown graph where end points of edges are selected uniformly at random we derived the generating function $g\left(x\right) = \sum_{k=1}^{\infty} k a_k x^k$ for the probability that a vertex selected at random is in a component of size k. Given that

$$g'(x) = \tfrac{1}{2\delta} \frac{1 - \frac{g(x)}{x}}{1 - g(x)}$$

derive the form of the average size of finite component as a function of the edge probability δ. You should determine the critical value of δ where the phase transition occurs and explain why a given solution is the correct solution for $\delta < \delta_{critical}$ and why a given solution is the correct solution for $\delta > \delta_{critical}$.

6. Small world graphs. Prove that for $r \geq 2$ there is always a short path. For what value of r do short paths cease to exist (open).

7. Small world graphs in 3-dimensions
(a) For what value or r is there an algorithm to find short paths?
(b) Prove for r=0 there is no local polylog time algorithm for finding short paths.

8. Small world 2-dimension. For r<2 could there be a polylog time local algorithm if the algorithm were allowed to explore random vertices at a cost of one per vertex explored. Make your model precise and then give a proof.