Homework 5 due Feb 27

Select five problems from the list below that interest you and solve them.

1. Examine the function $f(x) = 1 - e^{-dx} - x$ for d>1. Plot f(x) for several values of d. Mathematically derive the form of the function for $0 \le x$.

2. How would you generate G(n,p) on your laptop for $n = 10^6$ and $p = \frac{2}{n}$?

3. Explore the structure of $G\left(n, \frac{2}{n}\right)$. What is the degree distribution? What is the distribution of connected components? What is the number and size of biconnected components? If you remove the biconnected components from the giant component are there components that are connected to the core of the giant component by only two edges? How would you go about finding these?

4. Let S be the expected number of vertices discovered as a function of the number of steps t in a breadth first search of $G\left(n, \frac{d}{n}\right)$. Write a differential equation using expected values for the size of S. Show that the solution is $S = d\left(1 - \frac{S}{n}\right)$. Show that the normalized size *f* of the frontier is $f(x) = 1 - e^{-dx} - x$ where $x = \frac{t}{n}$ is the normalized time.

5. Prove that the expected value of the absolute value of the size of the frontier increases linearly with *i* for i in the neighborhood of Θ.

6. Consider the set of binomial distributions, binomial$\left[n-1, 1-\left(1-\frac{d}{n}\right)^i\right]$ for $d > 1$, one distribution for each value of i. Prove that as $n \to \infty$, the probability that a random variable from a distribution for a given i is zero goes to zero except for i in the two ranges $\left[0, \log n\right]$ and $\left[\Theta - \sqrt{n}, \Theta + \sqrt{n}\right]$.

7. The Molley Reed condition for the existence of a giant component is $\sum_{i=0}^{\infty} i(i-2)\lambda_i > 0$.

For the G(n,p), the probability that a vertex is of degree i is $\binom{n}{i} p^i (1-p)^{n-i}$. Show that the phase transition for a giant component occurs when $p = \frac{1}{n}$.

8. Explore the shape of the binomial distribution

$$P(k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}$$

for various values of p. Start with p=1/2 and decrease p down to 1/n.

Exercises on branching processes.

9. A branching process either dies out or goes to infinity. In biological systems there must be some other factor since processes seem to go to stable populations. One

possibility is that the probability distribution for the number of descendants of a child depends on the total population of the current generation.
Try to create a probability distribution which varies with the current population in which future generations neither die out nor grow to infinity.

10. Prove that

(a) $f_j\left(f(x)\right) = f\left(f_j(x)\right)$

(b) Prove that $p_1 + 2p_2 + 3p_3 + \cdots = 1$ and $p_1 \neq 1$ implies $p_0 > 0$. This corresponds to the case where $m = p_1 + 2p_2 + 3p_3 + \cdots = 1$ and $p_1 < 1$, and hence $p_0 > 0$.

11. Write a Matlab program and plot $f(x)$, $f_2(x)$, $f_3(x)$. What is $f_j(x)$ for very large j?

12. For each of the following probability distributions what is the extinction probability? the expected size assuming the branching process is finite?

a) $p_0 = \frac{1}{4}$ $p_1 = \frac{1}{2}$ $p_2 = \frac{1}{4}$

b) $p_0 = \frac{1}{8}$ $p_1 = \frac{3}{8}$ $p_2 = \frac{3}{8}$ $p_3 = \frac{1}{8}$

13. Explore different probability distributions. Can you summarize what happens in an interesting manner?

14. Can you create a branching process which gives rise to unbounded size finite sets? Clearly the probability distribution must be different for different vertices.

15. Think of the branching process as a Markov chain. For the case (0,1,2,3) having probabilities (1/3,0,1/3,1/3) what is stable probability distribution? If there is a drift to the right show that there is a probability of infinity. Is this true for all probability distributions or only for Poisson.

16. For $p_0 = p_2 = \frac{1}{2}$ prove that the expected size of extinct families is infinite.

Miscellaneous problems

17. Any problem that you make up.