Scribe Notes: CS 4850 : Math Foundations for the Information Age

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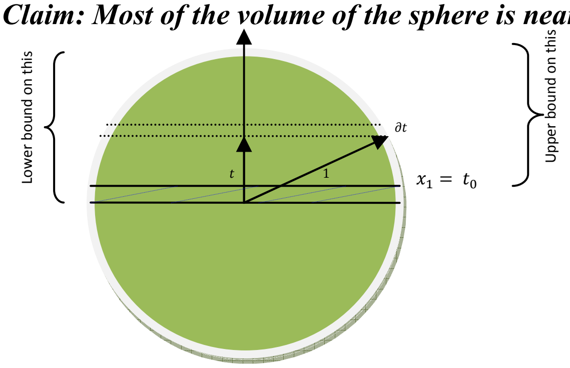
Volume of a sphere in high dimensions

Claim: Most of the volume of the sphere is near the equator.



Area of a $d - dimensional$ disk = volume of a $(d-1) - dimensional$ disk.

Volume of the slice between $x_1 = 0$ and $x_1 = t$ for the dimension x_1

Let T = $\{ \ x \mid |x| \leq 1; x_1 \geq t_0\}$

Radius of the slice, $r = \sqrt{1-t^2}$

Volume of the slice $= Surface\ Area\ \times radius = V(d-1)(1-t^2)^{\frac{d-1}{2}}\ dt$

Integrate this to get the total volume between t_0 and 1 : $\int_{t_0}^{1} V(d-1)(1-t^2)^{\frac{d-1}{2}}\ dt$

$$Vol(T) = \int_{t_0}^{1} V(d-1)(1-t^2)^{\frac{d-1}{2}}\ dt$$

$$= \ V(d-1) \int_{t_0}^{1} (1-t^2)^{\frac{d-1}{2}}\ dt$$

We know that $1 + x \leq \ e^x \ \forall\ real\ x.$

Therefore, $Vol(T) \leq V(d-1) \int_{t_0}^{1} e^{\frac{-t^2(d-1)}{2}}\ dt$

Since this integral drops off exponentially fast, replace 1 with ∞ to make the integral slightly larger.

$$\Rightarrow Vol(T) \le Vol(d-1) \int_{t_0}^{\infty} e^{\frac{-t^2(d-1)}{2}} dt$$

If we consider $\int_{t_0}^{\infty} e^{-\lambda t^2} dt \le \int_{t_0}^{\infty} \frac{t}{t_0} e^{-\lambda t^2} dt = \frac{1}{-2\lambda t_0} e^{-\lambda t^2} \Big]_{t_0}^{\infty} = \frac{1}{2\lambda t_0} e^{-\lambda t_0{}^2}$

(Done to give an upper bound since $\frac{t}{t_0} \ge 1$)

$$\Rightarrow Vol(T) \le Vol(d-1) \frac{1}{(d-1)t_0} e^{\frac{-t_0{}^2(d-1)}{2}}$$

Now calculating a lower bound in the hemisphere, H

$$Vol(H) = \int_0^1 V(d-1)(1-t^2)^{\frac{d-1}{2}} dt \ge V(d-1) \int_0^{\frac{1}{\sqrt{d-1}}} (1-t^2)^{\frac{d-1}{2}} dt$$

Applying the approximation, $(1-\epsilon)^m \ge (1-m\epsilon)$

$$Vol(H) \ge V(d-1) \int_0^{\frac{1}{\sqrt{d-1}}} (1 - \frac{d-1}{2} t^2) dt$$

Since $t \le \frac{1}{\sqrt{d-1}}$ $in\ the\ range\ 0\ to\ \frac{1}{\sqrt{d-1}}$

$$Vol(H) \ge \frac{Vol(d-1)}{t} \int_0^{\frac{1}{\sqrt{d-1}}} \frac{1}{2} dt = \frac{V(d-1)}{2\sqrt{d-1}}$$

Taking the ratio of the upper bound on the volume above the equatorial disk to the lower bound of the volume of the hemisphere:

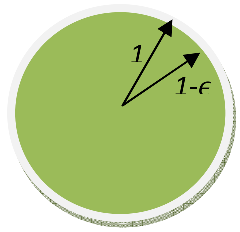
$$\frac{Vol(d-1)\frac{1}{(d-1)t_0} e^{\frac{-t_0{}^2(d-1)}{2}}}{\frac{V(d-1)}{2\sqrt{d-1}}} = \frac{1}{t_0\sqrt{d-1}} e^{-\frac{d-1}{2}t_0{}^2}$$

This ratio $\rightarrow 0$. Therefore, most of the volume of a sphere in high dimensions is near the equator.

Note: If $t_0 \rightarrow 0$, our approximations are not accurate enough.

Volume in Annulus

In d dimensions: $Vol\big(B(0,r)\big) = Vol\big(B(0,1)\big)\, r^d = Vol(d) r^d$



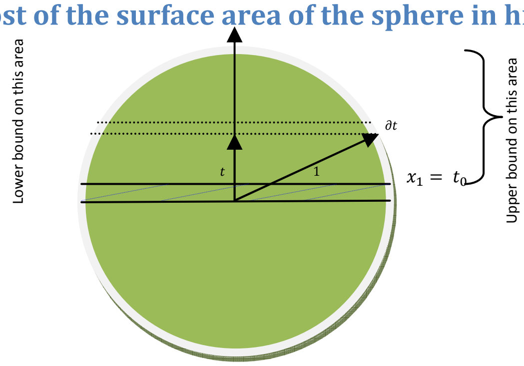
Put $r = (1-\epsilon) \Rightarrow Vol\big(B(0,r)\big) = Vol(d)(1-\epsilon)^d$

Now $(1+x) \leq e^x \Rightarrow (1-\epsilon) \leq e^{-\epsilon} \Rightarrow (1-\epsilon)^d \leq e^{-\epsilon d}$

$$\Rightarrow \boldsymbol{Vol\big(B(0,r)\big) \leq Vol(d)e^{-\epsilon d}}$$

Circumference of is a derivative of the surface area, which in turn, is a derivative of the volume.

Most of the surface area of the sphere in high dimensions is near the equator.



$$S = \{x \mid |x| = 1;\ x_1 \geq t_0\}$$

To calculate area: $Circumference \times thickness$

$$Area(S) = Vol(d-2)\int_{t_o}^{1}(1-t^2)^{\frac{d-2}{2}}dt$$

Proceeding exactly as we did above,

$$Area(S) \leq Vol(d-2)\frac{1}{(d-2)t_0}\ e^{\frac{-t_0{}^2(d-2)}{2}}$$

Lower bound on the area of the hemisphere = $\int_0^1 V(d-2)(1-t^2)^{\frac{d-2}{2}}dt$

$$\geq \frac{1}{2\sqrt{d-2}}V(d-2)$$

$$\frac{Upper\ bound\ on\ Area(S)}{Lower\ bound\ on\ Area(H)} \rightarrow 0$$

So most of the area of a sphere in high dimensions is near the equator.