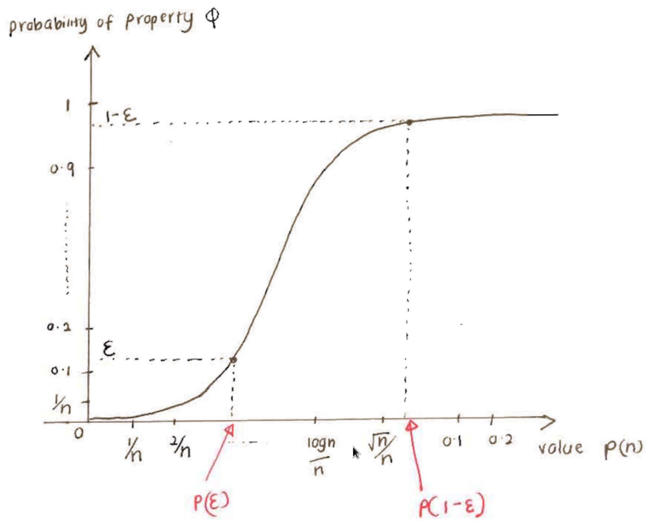
Increasing property: Q is an increasing property of a random graph G if when $p_1 < p_2$, $G(n, p_2)$ almost surely has Q if $G(n, p_1)$ almost surely has Q. i.e., if we increase the probability of an edge, the property Q becomes more likely.

Theorem: Every increasing property of $N(n, p)$ has a threshold. (The combinatorial structure does not actually matter.)

Proof: To show that $p(n)$ is a threshold for property Q, we need to show that the probability of property Q goes from 0 to 1 within a range that is bounded by a multiplicative constant.



We have to show that functions $p(\epsilon)$ and $p(1-\epsilon)$ are asymptotically equivalent. That is, we need to show that there exists as constant m such that $p(1-\epsilon) \leq mp(\epsilon)$.

Notational note: Let $p(n, \epsilon)$ be the function $p(n)$ such that the probability of Q is ϵ. We will also sometimes write N_p for $N(n, p)$.

Start of Proof

Let $0 < \epsilon < \frac{1}{2}$ and let m be an integer such that $(1-\epsilon)^m \leq \epsilon$. We now show that $p(1-\epsilon) \leq mp(\epsilon)$.

Take a number of independent copies of $N_{p(\epsilon)}$ and union them together. Consider the union of m independent copies of $N_{p(\epsilon)}$. The union is equivalent to $N(n, q(\epsilon))$, where:

$$q(\epsilon) = 1 - (1 - p(\epsilon))^m \leq mp(\epsilon)$$

Here, $1 - p(\epsilon)$ is the probability of not picking an integer in the original set, and $(1-p(\epsilon))^m$ is the probability of not picking an integer in any one of the sets.

Now we have that:

$$Prob(N_{mp(\epsilon)} \in Q) \geq Prob(N_{q(\epsilon)} \in Q) \tag{1}$$

If N_q does not have Q, then none of $N_{p(\epsilon)}$ have property Q. That is,

$$\begin{aligned} Prob(N_{q(\epsilon)} \notin Q) &= Prob(\forall N_{p(\epsilon)}, N_{p(\epsilon)} \notin Q) \\ &= (1 - Prob(N_{p(\epsilon)} \in Q)^m \\ &= (1-\epsilon)^m \leq \epsilon \end{aligned}$$

We also know that the probability that $N_{q(\epsilon)}$ has property Q is:

$$Prob(N_{q(\epsilon)} \in Q) \geq 1 - \epsilon \tag{2}$$

Combining Eqns. 1 and 2, we get:

$$Prob(N_{mp(\epsilon)} \in Q) \geq 1 - \epsilon$$

We must now argue that $mp(\epsilon) \geq p(1-\epsilon)$. $p(1-\epsilon)$ is the value of $p(n)$ such that $N_{p(n)}$ has Q with probability $1-\epsilon$, and since Q is an increasing property, we get $mp(\epsilon) \geq p(1-\epsilon)$. Now we're done for the following reasons.

$$p(\epsilon) \leq p\left(\frac{1}{2}\right) \leq p(1-\epsilon) \leq mp(\epsilon)$$

Thus, $p(\frac{1}{2})$ must be asymptotically equivalent to $p(\epsilon)$ and $p(1-\epsilon)$ is asymptotically the same as $p(\epsilon)$.

Example of non-monotonic property: Whether the number of elements in a set is even. This will vary non-monotonically as a single item is added to the set.

Another Combinatorial Structure - CNF boolean structure

Boolean formula of the form:

$$f(x) = (x_1 + x_2 + x_3)(\overline{x_1} + x_4 + x_5)(\dots)\dots(\dots)$$

Consider the case where we have n boolean variables and clauses with k variables in each clause. As the number of clauses increases, the probability of the formula having a satisfying assignment decreases. We seek to prove that this is a monotone property and therefore, there is a threshold. That is, there is some number of clauses above which the formula ceases to be satisfiable for any assignment of truth values to variables.

In other words: Generate a random CNF formula f with k literals per clause, n variables, and c clauses. Fix k and n. Then, the probability that f is satisfiable depends solely on the number of clauses.

Theorem: There exists a threshold, c where $c = r_k n$ and r is the ratio of number of clauses to number of variables.

First, let's think about this question: Given an assignment, what is the probability that a random clause is satisfiable? For a clause to be true, recall that only one of the k variables in the clause must be true. We know that the probability that a single clause of size k is satisfiable is $1 - \frac{1}{2^k}$. Suppose we have r_n independent clauses. **Then the probability that a random assignment satisfies the formula is:** $\mathbf{\left(1 - \frac{1}{2^k}\right)^{r_n}}$ **and the expected number of satisfying assignments is** $\mathbf{2^n \left(1 - \frac{1}{2^k}\right)^{r_n}}$**.**

What does r have to be to change the expected number of satisfying assignment from 0 to 1? Let $r = 2^k \ln 2$. Then we have:

$$\begin{aligned} 2^n\left(1 - \frac{1}{2^k}\right)^{2^k n \ln 2} &\approx 2^n e^{-n \ln 2} && \text{for reasonably large } k \\ &\approx 2^n \cdot 2^{-n} \\ &\approx 1 \end{aligned}$$