**Increasing property:** $Q$ is an increasing property of a random graph $G$ if when $p_1 < p_2$, $G(n, p_2)$ almost surely has $Q$ if $G(n, p_1)$ almost surely has $Q$. i.e., if we increase the probability of an edge, the property $Q$ becomes more likely.

**Theorem:** Every increasing property of $N(n, p)$ has a threshold. (The combinatorial structure does not actually matter.)

**Proof:** To show that $p(n)$ is a threshold for property $Q$, we need to show that the probability of property $Q$ goes from 0 to 1 within a range that is bounded by a multiplicative constant.

We have to show that functions $p(\epsilon)$ and $p(1 - \epsilon)$ are asymptotically equivalent. That is, we need to show that there exists as constant $m$ such that $p(1 - \epsilon) \leq mp(\epsilon)$.

**Notational note:** Let $p(n, \epsilon)$ be the function $p(n)$ such that the probability of $Q$ is $\epsilon$. We will also sometimes write $N_p$ for $N(n, p)$.

**Start of Proof**

Let $0 < \epsilon < \frac{1}{2}$ and let $m$ be an integer such that $(1 - \epsilon)^m \leq \epsilon$. We now show that $p(1 - \epsilon) \leq mp(\epsilon)$.

Take a number of independent copies of $N_p(\epsilon)$ and union them together. Consider the union of $m$ independent copies of $N_p(\epsilon)$. The union is equivalent to $N(n, q(\epsilon))$, where:

$$q(\epsilon) = 1 - (1 - p(\epsilon))^m \leq mp(\epsilon)$$

Here, $1 - p(\epsilon)$ is the probability of not picking an integer in the original set, and $(1 - p(\epsilon))^m$ is the probability of not picking an integer in any one of the sets.

Now we have that:

$$Prob(N_{mp(\epsilon)} \in Q) \geq Prob(N_{q(\epsilon)} \in Q) \tag{1}$$

If $N_q$ does not have $Q$, then none of $N_{p(\epsilon)}$ have property $Q$. That is,

$$Prob(N_{q(\epsilon)} \notin Q) = Prob(\forall N_{p(\epsilon)}, N_{p(\epsilon)} \notin Q)$$

$$= (1 - Prob(N_{p(\epsilon)} \in Q))^m$$

$$= (1 - \epsilon)^m \leq \epsilon$$

We also know that the probability that $N_{q(\epsilon)}$ has property $Q$ is:

$$Prob(N_{q(\epsilon)} \in Q) \geq 1 - \epsilon \tag{2}$$
Combining Eqns. 1 and 2, we get:

\[ \text{Prob}(N_{mp(e)} \in Q) \geq 1 - \epsilon \]

We must now argue that \( mp(e) \geq p(1 - \epsilon) \). \( p(1 - \epsilon) \) is the value of \( p(n) \) such that \( N_{p(n)} \) has \( Q \) with probability \( 1 - \epsilon \), and since \( Q \) is an increasing property, we get \( mp(e) \geq p(1 - \epsilon) \). Now we’re done for the following reasons.

\[ p(\epsilon) \leq p\left(\frac{1}{2}\right) \leq p(1 - \epsilon) \leq mp(\epsilon) \]

Thus, \( p\left(\frac{1}{2}\right) \) must be asymptotically equivalent to \( p(\epsilon) \) and \( p(1 - \epsilon) \) is asymptotically the same as \( p(\epsilon) \).

**Example of non-monotonic property:** Whether the number of elements in a set is even. This will vary non-monotonically as a single item is added to the set.

**Another Combinatorial Structure - CNF boolean structure**

Boolean formula of the form:

\[ f(x) = (x_1 + x_2 + x_3)(x_1 + x_4 + x_5)(\ldots)(\ldots) \]

Consider the case where we have \( n \) boolean variables and clauses with \( k \) variables in each clause. As the number of clauses increases, the probability of the formula having a satisfying assignment decreases. We seek to prove that this is a monotone property and therefore, there is a threshold. That is, there is some number of clauses above which the formula ceases to be satisfiable for any assignment of truth values to variables.

In other words: Generate a random CNF formula \( f \) with \( k \) literals per clause, \( n \) variables, and \( c \) clauses. Fix \( k \) and \( n \). Then, the probability that \( f \) is satisfiable depends solely on the number of clauses.

**Theorem:** There exists a threshold, \( c \) where \( c = r_k n \) and \( r \) is the ratio of number of clauses to number of variables.

First, let’s think about this question: Given an assignment, what is the probability that a random clause is satisfiable? For a clause to be true, recall that only one of the \( k \) variables in the clause must be true. We know that the probability that a single clause of size \( k \) is satisfiable is \( 1 - \frac{1}{2^k} \). Suppose we have \( r_n \) independent clauses. **Then the probability that a random clause is satisfiable is: \( (1 - \frac{1}{2^k})^{r_n} \)**, and the expected number of satisfying assignments is \( 2^n \left(1 - \frac{1}{2^k}\right)^{r_n} \).

What does \( r \) have to be to change the expected number of satisfying assignment from 0 to 1? Let \( r = 2^k \ln 2 \). Then we have:

\[
2^n \left(1 - \frac{1}{2^k}\right)^{2^k n \ln 2} \approx 2^n e^{-n \ln 2} \approx 2^n \cdot 2^{-n} \approx 1
\]

for reasonably large \( k \).