CS4850 Lecture Notes

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1 Threshold for graph connectivity

For a given p in $G(n, p)$, one of the following three things must be true:

1. graph is a single connected component
2. graph has an isolated vertex
3. graph has two or more components, but no isolated vertex

If we can show the probability of 3 is zero above the threshold for 1, then we know the threshod for 1 will be the same as the threshold for 2

Consider the probability of there being a component of size $k(n) = o(n)$, the neccessary conditions are:

- the $k(n)$ vertices must be connected
- no edges goes from one of the $k(n)$ vertices to some outside vertex

Let X be the number of components of size $k(n)$, then consider the expected value of X

$$
\begin{aligned}
E(X) &= \binom{n}{r}(1-p)^{nk-k^2}\delta \qquad \text{where } \delta < 1 \text{ is the prob that } k \text{ vertices connect} \\
&\leq \binom{n}{r}(1-p)^{nk-k^2} \\
&\simeq \binom{n}{r}(1-p)^{nk} \qquad \text{drop the } k^{@} \text{ since } k \text{ is } o(n)
\end{aligned}
$$

For $p = \frac{c \ln n}{n}$

$$
\begin{aligned}
E(X) &\leq \frac{n^k}{k!}(1 - \frac{c \ln n}{n})^{nk} \\
&= \frac{n^k}{k!} e^{-ck \ln n} \\
&= \frac{n^k}{k!} n^{-ck}
\end{aligned}
$$

For $c > 1$, $\frac{n^k}{k!} n^{-ck} = \frac{n^{(1-c)k}}{k!} \to 0$, so almost surely no component of size $o(n)$.

2 Phase Transitions

Define $N(n,p)$ subset of integers from $\{1,\ldots,n\}$ where each integer gets selected with probability p.

We are interested in the property: The set contains an arithmetic progress of length k.

$$a, a+b, \ldots, a+(k-1)b$$

Let X_k be the number of arithmetic progression of length k.

$$p = n^{-\frac{2}{k}} \text{ is the threshold}$$

Consider the expected value of X_k,
An arithmetic progression is determined by its starting value and the interval, and both of the two factors have roughly n choices, thus the total number of arithmetic progression is roughly n^2, the probability that all the k numbers of the progression are in the set is p^k, thus at $p = n^{-\frac{2}{k}}$

$$\begin{aligned} E(x_k) &= n^2 p^k \\ &= n^2 n^{-k\frac{2}{k}} \\ &= 1 \end{aligned}$$

It is obvious to see $E(X_k) \to 0$ if $p < n^{-\frac{2}{k}}$, and $E(X_k) \to \infty$ if $p > n^{-\frac{2}{k}}$

We know $E(X_k) = 0$ suggests $Prob(\text{arith. progres. of length k}) = 0$ by the same argument from last lecture.

When $E(X_k) = \infty$, we need to apply second moment method.
Want to show

$$\frac{E(X_k^2)}{E^2(X_k)} \to 1$$

Denote random variables $I_1, \ldots$ such that

$$I_j = \left\{ \begin{array}{ll} 0 & \text{if } j\text{th arith. progress. is not present} \\ 1 & \text{if } j\text{th arith. progress. is present} \end{array} \right.$$

Then $X_k = I_1 + I_2 + \dots$

$$\frac{E(\sum_{i,j} I_i I_j)}{[E(\sum_i I_i)][E(\sum_j I_j)]} \tag{1}$$

$$\leq \frac{n^4 p^{2k} + n^3 k p^{2k-1} + n^2 k^2 p^{2k-2}}{(n^2 p^k)(n^2 p^k)} \tag{2}$$

$$= 1 + \frac{k}{np} + \frac{k^2}{n^2 p^2} \tag{3}$$

$$\to 1 \tag{4}$$

The terms in the numerator of step 2 correspond to pairs of arith. progress sharing no numbers, pairs of arith. progress sharing one numbers, and so on.

By second moment argument, we know
$Prob(\text{arith. progress.}) = 1$ when $E(X_k) = \infty$.