Properties of G(n,p)
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Sara Tansey(sjt33) and Yuzhe Liu(yl435)

1 Phase Transitions

- $p = 0$ isolated vertices
- $p = \Theta(1/n)$cycles appear
- $p = 1/n$ giant component appears
- $p = (1/4)log(n)/n$ no small components (only giant and single)
- $p = log(n)/n$ first achieve connectivity
- $p = aconstant$ diameter becomes 2

a *phase transition* is:
- given a threshold $p(n)$ if we have two functions $p_1(n)$ and $p_2(n)$ where $p_1(n)$ is asymptotically slower and $p_2(n)$ is asymptotically faster than $p(n)$ then we know that $G(n, p_1(n))$ does not display the property and $G(n, p_2(n))$ will display the property.

a *sharp phase transition* is:
- given a threshold $c * p(n)$ if $c < 1$ then the graph does exhibit the property and if $c > 1$ the graph does not exhibit the property

A giant component will have a constant size $(c * n)$ as $n \to \infty$.

2 Disappearance of Isolated Vertices

let x be the number of isolated vertices
$E(x) = n(1-p)^{n-1}$ as n approaches infinity
This is the number of vertices multiplied by the probability that a vertex does not have edges to the other $n-1$ vertices.

let $P = (c \log n)/n$

$$\begin{aligned}
E(x) &= n(1 - (\frac{(c \log n)}{n}))^{\frac{n * c \log n}{(c \log n)}} \\
&= n * (1/e)^{\frac{1}{c \log n}} \\
&= ne^{-c \log n} \\
&= nn^{-c} \\
&= n^{1-c}
\end{aligned}$$

So we see if $c < 1$, $E(x)$ will be n to a positive exponent. $E(x) \to \infty$
If $c > 1$, $E(x)$ will be n to a negative exponent. $E(x) \to 0$

Given this information, what can we say about the probability that a given graph will have isolated vertices?

If $c > 1$ and $E(x) = 0$ as $x \to \infty$, there cannot be some constant fraction of graphs that contain isolated vertices, because then the expected number of isolated vertices would be nonzero. So the probability that a given graph has isolated vertices is 0.

If $c < 1$ and $E(x) = \infty$ as $x \to \infty$, how do we know that it isn't the case that half of the graphs have 0 isolated vertices and half have an infinite number? We will show that this is unlikely by showing that the probability distribution of the number of isolated vertices is centered tightly.

3 Second Moment Argument

We will use this a lot.

3.1 Markov Inequality

we have a random variable $x \geq 0$ and we want to figure out what is the probability that x is twice or three times its expected value (see Figure 1): $P(x \geq 2*E(x)$ or $P(x \geq 3*E(x)$.

In the worst case, what is the highest probability that $x \geq 2*E(x)$ (see figure 2)? In the worst case, what is the highest probability that $x \geq 3*E(x)$ (see figure 3)?

3.1.1 Proof:

that $Prob(x \geq a) \leq E(x)/a$

$$\begin{aligned} E(x) &= \int_0^\infty x * p(x)\,dx. \\ &= \int_0^a x * p(x)\,dx. + \int_a^\infty x * p(x)\,dx. \\ &\geq \int_a^\infty x * p(x)\,dx. \end{aligned}$$

since $x >= a$:

$$\begin{aligned} E(x) &\geq \int_a^\infty a * p(x)\,dx. \\ &= a \int_a^\infty p(x)\,dx. \\ &= Prob(x \geq a) \end{aligned}$$

By rearranging we get: $Prob(x \geq a) \leq \frac{E(x)}{a}$
And a Corollary: $Prop(x \geq a * E(x) \leq 1/a$

Figure 1: Figure 1

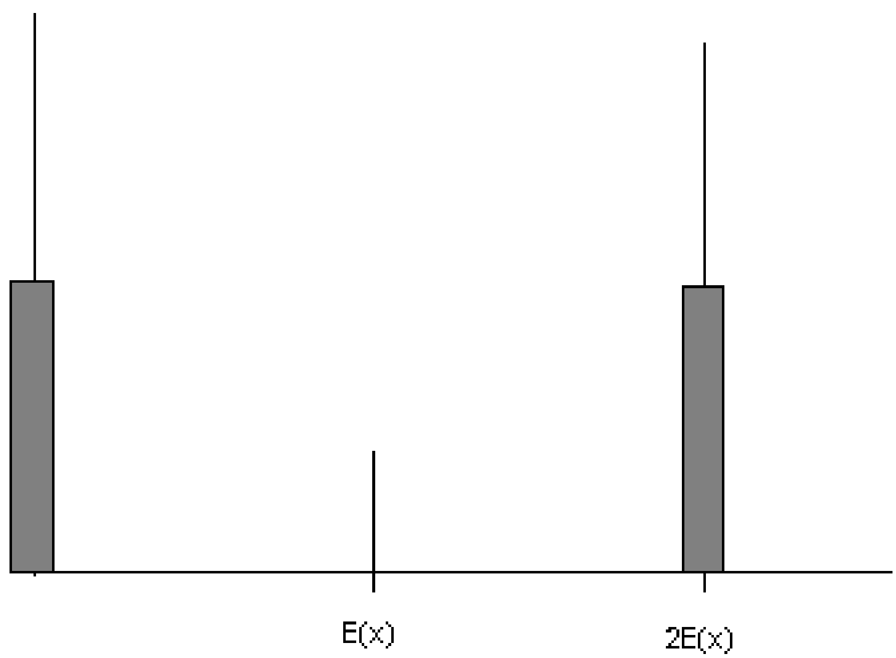

Figure 2: Figure 2

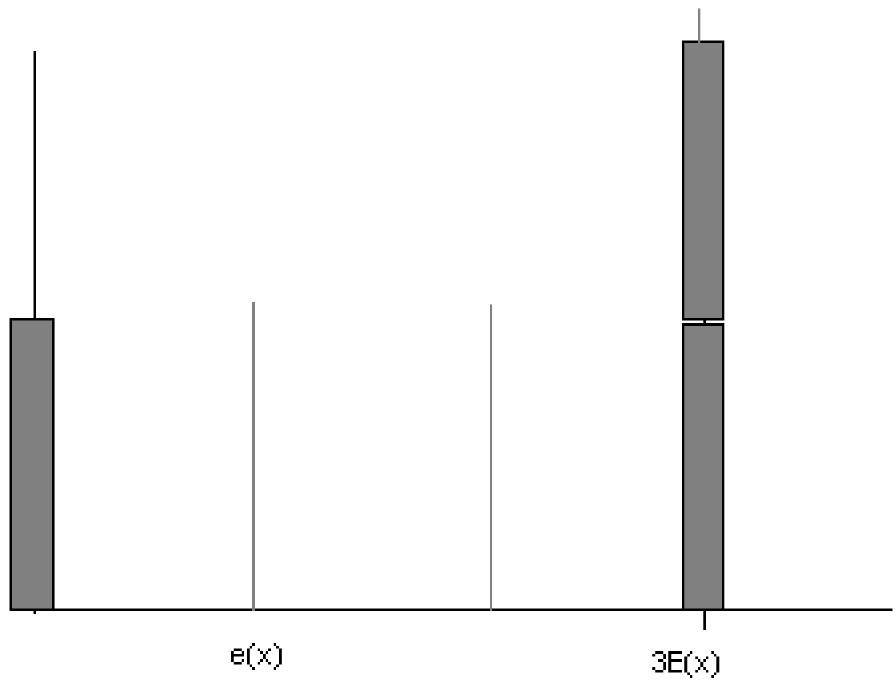

Figure 3: Figure 3

3.2 Chebyshev's Inequality

We might be able to make a stronger claim if we said something about the variance.

3.2.1 Theorem

Let x be a random variable with mean m and variance σ squared.
Then $Prob(|x-m| \geq a\sigma) \leq 1/a^2$

3.2.2 Proof

$$\begin{aligned} Prob(|x-m| &\geq a\sigma) \\ &= Prob((x-m)^2 \\ &\geq a^2\sigma^2) \\ &\leq \frac{E((x-m)^2)}{(a^2\sigma^2)}, \end{aligned}$$

To get the last line, we used Markov's inequality.
Thus, $Prob(|x-m| \geq a\sigma) \leq 1/a^2$.

3.3 Independence

If random variable x is made of statistically independent variables $x_1 + x_2 + ...$ then we can give an even tighter bound. More on this later.

3.4 Second Moment Argument

If the expected value is large with respect to variance, the random variable takes value 0 with probability 0. If $E(x) >> \sigma^2$, $Prob(x=0) = 0$.

3.5 Proof

Start out with:

$$Prob(x=0) \leq Prob(|x-E(x)| \geq E(x))$$

By Chebyshev (let $E(x) = a\sigma -> 1/a = \sigma/E(x)$):

$$\begin{aligned} Prob(x=0) &\leq \sigma^2/E^2(x) \\ &= \frac{E((x-E(x))^2)}{/}E^2(x) \\ &= \frac{E(x^2 - 2xE(x) + E^2(x))}{E^2(x)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{E(x^2) - 2E^2(x) + E^2(x)}{E^2(x)} \\
&= \frac{(E(x^2) - E^2(x))}{E^2(x)} \\
&= \frac{E(x^2)}{E^2(x)} - 1
\end{aligned}
$$