CS 485: Mathematical Foundations for the Information Age

Lecture 32 ▪ April 08, 2009

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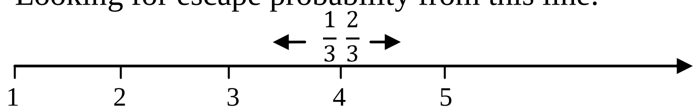
From HW:

Looking for escape probability from this line:



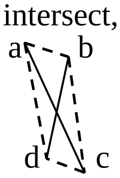
Similar problem: effective resistance of binary tree – equation here is $R_{eff} = ½(1+R_{eff})$, which says that $R_{eff} = ½$. However, when we solve the recurrence relation with the line, we get an equation of the form $Pr(Escape) = A+B(½)^i$. What should the boundary conditions be?

Review:

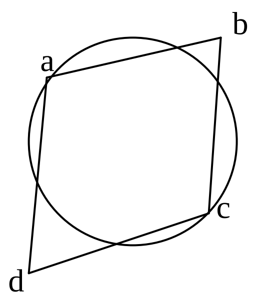
- Set systems: (X, $\mathcal{S}$) – X is a set, $\mathcal{S}$ is a collection of subsets (i.e. shapes)
- A set A can be *shattered* if every subset of A can be expressed as A ∩ S, S in $\mathcal{S}$.
- *VC dimension:* cardinality of largest set that can be shattered by a class of subsets.

Goal – VC dimension with circles

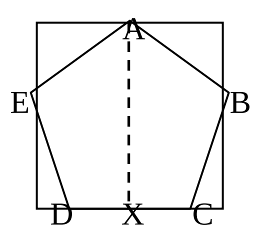
- Will need this – Radon's Theorem (http://en.wikipedia.org/wiki/Radon's_theorem):
    - Convex hull: the convex polygon (or higher-dimensional analog) containing all of the points of a set.
    - Given d+2 points in d dimensions, the points can always be partitioned into two sets A and B where the convex hulls of A and B have a non-empty intersection, i.e., convex(A) ∩ convex(B) ≠ {}.
    - To prove this in general: form a regular shape with d+1 vertices. Then there will be d cases for where the final vertex can go – either it will be on the inside, or in one of the regions outside of the shape, for which there d-1 different cases depending on the face of intersection. Pick A to be the face of intersection, and B to be the remaining points, and they will have at least one point in common.
- VC dimension of circle
    - Showed last lecture – can shatter 3 points. Want to prove it is impossible to shatter 4.
    - By Radon's Theorem, partition the 4 points into sets A and B. If either A or B has 3 points, we are done, because the set with 3 points must be a triangle, so the last point must be inside the triangle, so we know we can't pick the 3 points without picking the point inside the triangle.
    - Otherwise, we label the vertices a,b,c,d, where convex({a,c}) and convex({b,d}) intersect, like so:



        - Either <a + <c ≥ 180° or <b + <d ≥ 180°. Without loss of generality, assume that <b + <d ≥ 180°. Suppose we managed to enclose a and c in a circle without including b and d. We want to prove this leads to a contradiction. So, we take the circle around a and c and start to shrink the radius until we hit a point on the boundary. Clearly, no new points have entered the circle.
        - Now, we must both shrink the radius and move the center towards the first point to avoid adding new points to the circle, until we have both a and c on the boundary (open problem – can we continue shrinking until we get 3 points on the boundary, without enveloping points outside of the circle?):



        - If b and d were on the circle, then <b + <d = 180° because opposite angles of a quadrilateral in a circle sum to 180. However, b and d must be outside the circle by our assumption, which clearly decreases the sum of the angle. So, <b + <d ≤ 180°.
        - This is a contradiction! Thus, there is no circle containing {a,c} and not containing {b,d}, so no set of four points can be shattered by circles, and the VC dimension of circular subsets is 3.
    - VC dimension of squares that can be rotated
        - Without rotation – VC dimension is 3 (previous lecture)
        - Adding rotation – can shatter 5 points. Prof. Hopcroft does not believe we can shatter 6 points, but this is an open problem – see HW.
        - Draw regular pentagon:



        - Trivial cases – {}, 1 element, 4/5 elements, 2/3 adjacent elements
        - By symmetry – show we can enclose {A,C} and {A,C,D}.
            - {A,C,D} – we know |AX| < |AC| = |BE|, so B and E are not included in the above square.
            - {A,C} – rotate this square slightly to the left, pivoted at A – we lose D but retain C.
    - Next time – more with Radon's Theorem.