CS4850 Math Foundations for the Information Age

Lecture 30

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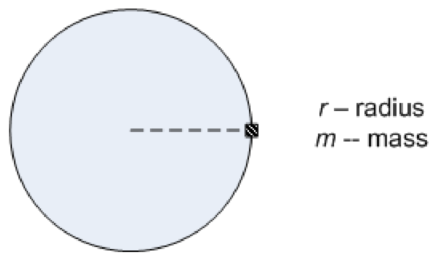
“Moment”: integral

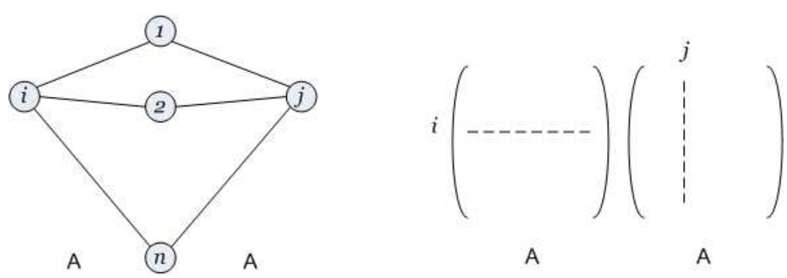


$$\text{inertia} \;=\; \int r^2 dm$$

$$r\text{th moment} \;=\; E[(x-m)^r] \;=\; \int (x-m)^r p(x)dx$$

where m is the mean and the variance is the second moment. All finite moments of a function fully define the function.

“Paths and graphs”: adjacency matrix



Matrix Entries	Operations	Interpretation
$0, 1$	AND, OR	transitive closure
distance	sum, min	shortest path
labels	concatenation, union	set of all paths

Given a random matrix A with entries belonging to $\{-1, 1\}$, each entry can be considered as the label of the corresponding edge. A path can thus be represented by the product of the labels of the edges along the path. Further, a set of paths can be represented by the sum of the labels of the paths within the set.

Specifically, $\left(A^k\right)_{ij}$ corresponds to the set of all paths of (exact) length k between i and j, which is a random variable. Each entry in A^k is the sum of the labels of all paths of length k, and each path is the product of the labels (1 or -1) along that path.

$$E\left(A_{ij}\right) = 0 \qquad\qquad E\left(A_{ij}^2\right) = 1$$

$$E\left(\lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k\right) = E\left(trace\left(A^k\right)\right)$$

The kth moment of normalized eigenvalues is given by:

$$m(k) = \frac{1}{2^k}\,\frac{1}{n^{1+k/2}}\;E\left(trace\left(A^k\right)\right)$$

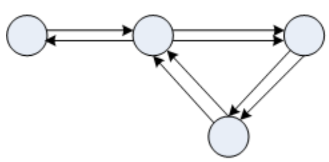
To compute the expected value of $\left(A^k\right)_{ii}$ $(1 \leqslant i \leqslant n)$,

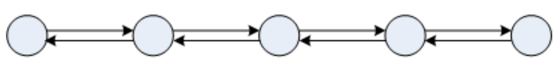
1. Each edge in the path appears at least twice.

   Note that each edge is labeled independently. If an edge (i,j) is traversed only once along some path, the label for that path is 0, since $E\left(A_{ij}\right) = 0$. Thus, we can ignore such a path.

2. We only need to consider paths with $k/2$ vertices.

   The number of ways to embed a path of length k with less than $k/2$ vertices is of lower order than the number of ways to embed a path of length k with $k/2$ vertices, for example,

   As $n \to \infty$, the paths of length k with less than $k/2$ vertices can be ignored.

Therefore,

$$m(k) \;=\; \frac{1}{2^k}\,\frac{1}{n^{1+k/2}} \cdot n \cdot n^{k/2} \cdot \mathrm{catalan}\,(k/2)$$

where n is the number of diagonal elements in matrix A, $n^{k/2}$ is the number of ways to embed a certain type of graph, and catalan$(k/2)$ is the number of shapes of the DFS trees.

$$\mathrm{catalan}(k/2) \;\triangleq\; C(k/2) \;=\; \frac{1}{1+k/2}\binom{k}{k/2}$$

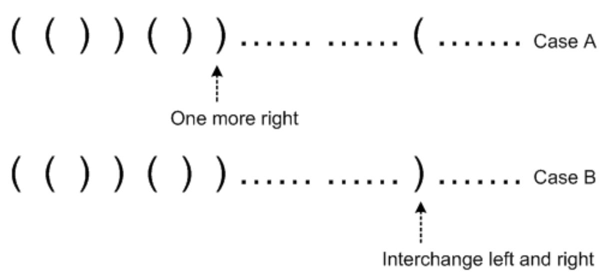
$$m(k) \;=\; \frac{1}{2^{k-1}}\,\frac{1}{k+2}\,\binom{k}{k/2}$$

Catalan numbers are the number of strings of length $2n$ balanced parentheses.

1. The number of strings of length $2n$ with equal number of left and right parentheses is given by:

$$\binom{2n}{n}$$

2. Each of these strings is balanced unless there is a prefix with one more right parentheses than left parentheses, as shown below:



There is one-to-one correspondence between strings with equal number of left and right parentheses but not balanced (Case A) and strings of length $2n$ with $n-1$ left parentheses (Case B).

$$\text{case A} \rightleftharpoons \text{case B}$$

The number of strings of length $2n$ with $n-1$ left parentheses is given by:

$$\binom{2n}{n-1}$$

Therefore,

$$\begin{aligned}
C(n) &= \binom{2n}{n} - \binom{2n}{n-1} \\
&= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{(2n)!}{n!(n+1)!} \\
&= \frac{1}{n+1}\,\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}
\end{aligned}$$