CS4850 - Lecture #29 Notes

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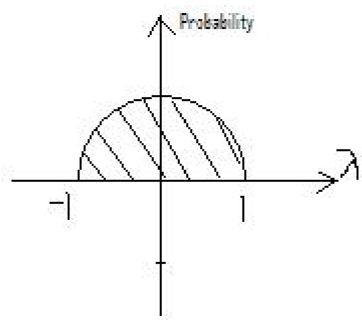
Eigenvalues of a Random Graph

Let A be a symmetric matrix whose elements are independent identically distributed random variables from a symmetric probability dist. With var = σ^2 & all of whose moments are finite.

Special case: $a_{ij} \in \{-1, 1\}$ => eigenvalues of A are in range: $[-2\sqrt{n}, 2\sqrt{n}]$



Normalize eigenvalues to [-1, 1]

To show: $P(\lambda) = \left(\frac{2}{\pi}\right)\sqrt{1-\lambda^2}$

Let c(k) be the kth moment of P(λ)

$c(k) = \left(\frac{2}{\pi}\right)\int_{-1}^{1}\lambda^k * \sqrt{1-\lambda^2}\,d\lambda$

Note if k is odd, the function is odd, so the integral is 0.

If k is even, substitute $\lambda = \sin\theta$.

So c(k) = $\left(\frac{2}{\pi}\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin\theta)^k * (\cos\theta)^2 \, d\theta$

= $\left(\frac{2}{\pi}\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin\theta)^k \, d\theta \; - \; \left(\frac{2}{\pi}\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin\theta)^{k+2} \, d\theta$

Note: $\int (\sin\theta)^n d\theta = -\frac{(\ (\sin\theta)^{n-1}\cos\theta\ )}{n} + \frac{n-1}{n} \int (\sin\theta)^{n-2} d\theta$

cos(+/- $\frac{\pi}{2}$) = 0

$$\left(\frac{2}{\pi}\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin\theta)^n \, d\theta = \frac{n-1}{n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin\theta)^{n-2} \, d\theta$$

Rinse & repeat until we go down to sin(θ)

... = $\frac{n-1}{n} * \frac{n-3}{n} * \ldots * \frac{1}{2}\pi$

$$C(k) = 2 * \frac{\left(1 * 3 * 5 * \ldots * (k-1)\right)}{2 * 4 * \ \ldots * k} - 2 * \frac{\left(1 * 3 * 5 * \ldots * (k+1)\right)}{2 * 4 * \ \ldots * (k+2)}$$

$$= 2 * \frac{\left(1 * 3 * 5 * \ldots * (k-1)\right)}{2 * 4 * \ \ldots * k * (k+2)} = 2 * \frac{k!}{(2 * 4 * \ \ldots * k)^2 * (k+2)}$$

$$= 2 * \frac{k!}{2^k * \left(1 * 2 * 3 * \ldots * \frac{k}{2}\right)^2 * (k+2)}$$

(From factoring out the 2s in the denominator's squared expression)

$$= \frac{k!}{\left(\frac{k!}{2}\right)^2} * \frac{1}{(k+2)2^{k-1}} = \binom{k}{k/2} * \left(\frac{1}{(k+2) * 2^{k-1}}\right)$$

…which is the kth moment of the function $\left(\frac{2}{\pi}\right)\sqrt{1-\lambda^2}$.

Let m(k) be moments of normalized eigenvalues:

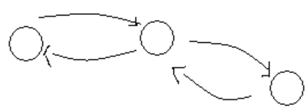
M(k) = $E[1/n * (\lambda_j/(2*\text{sqrt}(n))^k] = 1/(2^k n^{(1+k/2)})E[\Sigma\lambda_j^k] =$ $1/(2^k n^{(1+k/2)})E[\text{Trace}(A^k)]$

Recall: $\text{Tr}(A) = \Sigma\, a_{ii} = \Sigma\lambda_j$ and $\text{Tr}(A^k) = \Sigma(\lambda_j^k)$

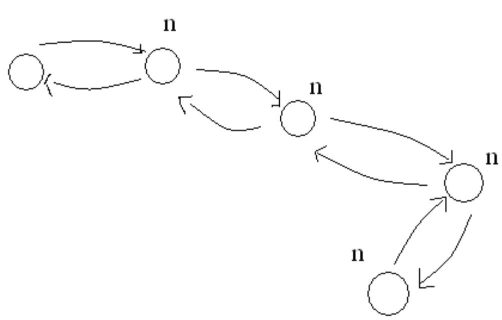
Note: The ij-th entry of A indictates edge from vertex i to vertex j. $(A^k)_{ij}$ = sum of all paths of length k from I to j.

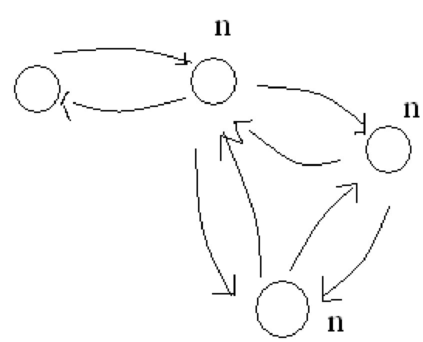
To find the expected value of a path, we can disregard a path that travels through a particular edge only once, since the value of the edge is a random variable of either -1 or 1, and E[path]=E[edge]*E[edge]*….*E[edge], if one of the edges has expected value of zero (which is the case when we only go through it once, or any odd number of times for that matter), the whole path will have an expected value of 0.

So we only want to consider paths that go through an edge twice, such as:



Note that for each vertex, we can get n possible embeddings, all independent of each other, so a path with k nodes (excluding the beginning node, which is fixed), we will have n^k possible embeddings.





Paths of length k=8 with n^4 embeddings and only n^3 embeddings

As a result, we don't have to consider paths of length k that has less than k/2 vertices, because those paths have less degree than the ones that do have k/2 vertices, and as n goes to infinity, they no longer matter.

These paths can be represented by a depth first search tree, which is related to the Catalan numbers, which will be continued in the next lecture.