Lecture 28: Matrix Norms and Eigen values

Matrix Norms

Frobenius norm $\sqrt{\sum_{i,j} a^2_{ij}}$

2-norm $\sigma_j^2 u_j$ $\max_{|v|=1} |Av| = \sigma_1$

$|A|_F = |U\Sigma V^T|_F = |\Sigma|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \ldots}$

$|A|_2 = |U\Sigma V^T|_2 = |\Sigma|_2 = \sigma_1$

Theorem:

$$(-1)^n(\lambda^n - (a_{11} + a_{22} + \ldots + a_{nn})\lambda^{n-1} + \ldots)$$

$|A - A_k|_F = (-1)^n(a_{11} + a_{22} + \ldots + a_{nn}) = (-1)^n(\lambda_{11} + \lambda_{22} + \ldots + \lambda_{nn})$ $\min_{rank(B)=k} |A-B|_F$

$$c(k) = \frac{2}{\pi}\int \lambda^k \sqrt{1-\lambda^2}\, d\lambda$$

$|A - A_2|_2 = \min_{rank(B)=k} |A-B|_2$

Let B be a square matrix. Find solutions of the form: B x = λx

Consider B = $AA^T = (\sum_{i=1}^{n} \sigma_i u_i v_i^T)(\sum_{j=1}^{n} \sigma_j u_j v_j^T)$

$= \sum_{i,j} \sigma_i \sigma_j u_i v_i^T v_j u_j^T = \sum_i \sigma_i^2 u_i u_i^T$

$Bu_j = \sum_i \sigma_i^2 u_i u_i^T u_j = \sigma_j^2 u_j$ $\quad\quad\quad * u_i u_i^T = 1$ when $i = j$

B is symmetric and positive semi definite

$x^T Bx \geq 0 \quad\quad x^T AA^T x = (A^T x)^2 \geq 0$

What does the probability distribution of Eigen values of random matrix look like? (Wigner, 1957)
The distribution of eigenvectors can tell you whether a matrix is random.

Lemma:

trace(A) = $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n$

trace(A^2) = $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \ldots + \lambda_n^2$

Proof: There exists a nontrivial solution for x in $(A - \lambda I)x = 0$ only if $\det(A - \lambda I) = 0$
$\det(A - \lambda I)$ is polynomial in λ.

$$(-1)^n \prod_{i=1}^{n} (\lambda - \lambda_i) = (-1)^n(\lambda^n - (\lambda_1 + \lambda_2 + \ldots + \lambda_n)\lambda^{n-1} + \ldots)$$

$$\det \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots \\ a_{21} & a_{22} - \lambda & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

= $(a_{11} - \lambda)\det(n - 1) + \ldots$
= $(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)\ldots(a_{nn} - \lambda) + \ldots$
= $(-1)^n(\lambda^n - (a_{11} + a_{22} + \ldots + a_{nn})\lambda^{n-1} + \ldots)$

We want to equate λ^{n-1} coefficients. The coefficient for λ^{n-1} is $(-1)^n(a_{11} + a_{22} + \ldots + a_{nn})$

$$(-1)^n(a_{11} + a_{22} + \ldots + a_{nn}) = (-1)^n(\lambda_{11} + \lambda_{22} + \ldots + \lambda_{nn})$$

We want to show P(λ) is like $\sqrt{1-\lambda^2}$ (semicircle)
Normalize the Eigen values to range [-1, 1]

$$\frac{2}{\pi}\sqrt{1-\lambda^2}$$

Find equal moments for both equations to show equality
K[th] moment:

$$c(k) = \frac{2}{\pi}\int \lambda^k \sqrt{1-\lambda^2}\, d\lambda$$