CS 4850 LECTURE 23 - MARCH 11, 2009

SCRIBES: YU-KANG CHENG (YC362) AND SARA TANSEY (SJT33)

Random Walks and Hitting Time

Where is the time spent?
Let $t(i)$ be the expected time spent at a vertex i on walk from vertex 1 to n.

$$t(n) = 1, t(n-1) = 2,$$
$$t(i) = \frac{1}{2}(t(i-1) + t(i+1))\,\forall\, i, 3 \leq i \leq n-2$$
$$t(2) = t(1) + \frac{1}{2}t(3), t(1) = 1 + \frac{1}{2}t(2)$$

An Interlude into Differential Equations
Given we want to solve the differential equation

$$\frac{d^3}{dx^3}f + 2\frac{d^2}{dx^2}f + \ldots = 0$$

We assume the form of all solutions is $f = e^{ax}$ for some a.
Plugging into the equation, we get:

$$a^3e^{ax} + 2a^2e^{ax} + \ldots = a^3 + 2a^2 + \ldots = 0$$

since e^{ax} cannot equal 0. This is called the characteristic equation. Additionally, all solutions of the form $f = e^{ax}$ are linearly independent for different values of a, so the general solution is

$$f = Ae^{ax} + Be^{bx} + Ce^{cx}$$

If multiple (m) roots occur at the same value, then take derivatives:

$$Ae^{ax} + A'xe^{ax} + \ldots + A^{(m-1)}x^{m-1}e^{ax}$$

The same rules apply to difference equations, which is what $t(i)$ is. Rewrite $t(i)$ as

$$t(i+1) - 2t(i) + t(i-1) = 0$$
$$a^2 - 2a + 1 = (a-1)^2 = 0$$

The characteristic equation factors, and thus the solution is $a = 1$.
Thus, $t(i) = c + di$
From the definition of $t(n-1)$:

$$2 = t(n-1) = \frac{1}{2}t(n-2) \Rightarrow t(n-2) = 4$$

From the definition of $t(n-2)$:

$$4 = t(n-2) = \frac{1}{2}(t(n-1) + t(n-3)) = \frac{1}{2}(2 + t(n-3)) \Rightarrow t(n-3) = 6$$

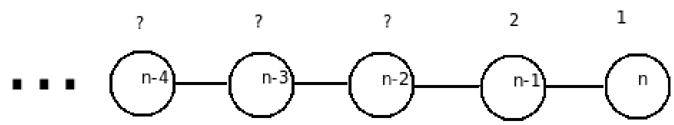

FIGURE 1. $t(i) = 2n - 2i$

This argument can be used recursively to show that $t(i) = 2n - 2i \, \forall \, i, 3 \leq i \leq n - 1$
For the lower boundary, $t(3) = 2n - 6$ from the definition. Then,

$$t(2) = 2n - 4, t(1) = 2n - 2$$

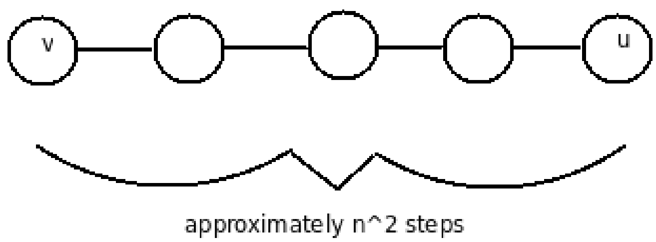

FIGURE 2. n^2 steps

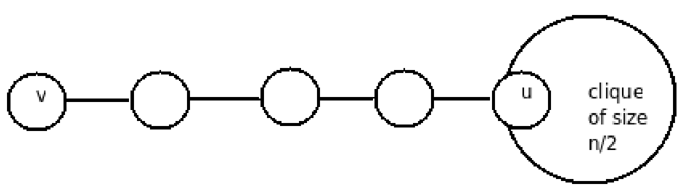

FIGURE 3. n^3 steps

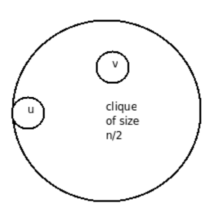

FIGURE 4. n steps

What is the maximum hitting time?
Lemma
If u and v are connected by an edge, then $h_{uv} + h_{vu} \leq 2m$, where m is the number of edges.
Proof

$$Pr(\text{traverse edge in one direction}) = Pr(\text{Select starting vertex}) * Pr(\text{Selecting edge})$$
$$= \frac{\text{deg}}{2m} \frac{1}{deg} = \frac{1}{2m}$$

Thus we see that the probability of traversing an edge is independent of the degrees of the vertices
Then, the expected time between traversals of an edge in a particular direction is $\frac{1}{\text{Prob}} = 2m$.
A random walk has no memory. After traversing edge uv, in approximately $2m$ steps we traverse the same edge again. Regardless of what edge we use to get to v, if we are at v, it would be the same amount of time ($2m$) before we traverse edge uv.

To Be Continued ...