Random Walk in 3-Dimensions
CS 4850 Notes – Lecture 22, Monday, March 9, 2009
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**Review**

1. \( v_x \) is probability that a walk starting at \( x \) reaches \( a \) before \( b \).
   
   \[ v_a = 1 \quad v_b = 0 \]

2. \( i_{xy} \) is net traversals of edge \( xy \) in walk from \( a \) to \( b \).

3. \( P_{\text{escape}} = \frac{c_{\text{eff}}}{c_a} \)

**1D**

Bend line: \( a \quad b \)

\( \lim_{n \to \infty} \) will return to \( a \) with probability 1.

**2D**

Many parallel paths but resistance is still infinite.

**3D**

Start with 2D, then will generalize to 3D.
Upper bound on $R_{eff}$

$$R = \frac{1}{2} + \frac{1}{4}(2) + \frac{1}{8}(4) + \ldots = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots = \infty$$

$$\frac{1}{2} \left[ 1 + \frac{1}{2} \left[ 2 + \frac{1}{2}(4 + R) \right] \right]$$

$$= \frac{1}{2} + \frac{1}{4}(2) + \frac{1}{8}(4) + \frac{1}{16}(R)$$

$$x + y + z = 2^k - 1$$
\[ P_{\text{escape}} \geq \frac{1}{6} \]

\((c_a = 6 \text{ because there are 6 edges coming out of } a \text{ if you consider more than just 1 quadrant})\)

**Lower bound on \( R_{\text{eff}} \)**

2D:
\[
\frac{1}{4} \left( 1 + \frac{1}{3} + \frac{1}{5} + \ldots \right)
\]

3D:
\[
R_{\text{lower bound}} = \frac{1}{6} \left[ 1 + \frac{1}{9} + \frac{1}{25} + \ldots \right] \geq \frac{1.23}{6} \geq 0.2
\]

\[
P_{\text{escape}} = \frac{1}{6} R_{\text{eff}} \leq \frac{5}{6}
\]
Page Rank and Hitting Time

*Directed graphs*
Stationary probability = page rank

Hitting time $h_{uv} = \text{expected time to reach } v \text{ from } u.$

Generalize – personalized page rank and personalized hitting time using probability distribution

*Undirected graphs*
If I add an edge to graph, do I raise or lower hitting time?

Hitting time is not symmetric:

$h_{uv} = 1$

$h_{vu} > 1$
Lemma: Expected time of random walk starting at one end of a chain of \( n \) vertices to reach the other end is \( \Theta(n^2) \).

\[
\begin{array}{cccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow \cdots \rightarrow \ n-1 & \rightarrow \ n
\end{array}
\]

\[
h_{ij} \quad i < j
\]

\[
h_{12} = 1
\]

\[
h_{i,i+1} = \frac{1}{2}(1) + \frac{1}{2}[1 + h_{i-1,i} + h_{i,i+1}]
\]

\[
\frac{1}{2}h_{i,i+1} = 1 + \frac{1}{2}h_{i-1,i}
\]

\[
h_{i,i+1} = h_{i-1,i} + 2
\]

\[
h_{i,i+1} = 2(i - 1) + h_{12} = 2i - 1
\]

\[
h_{1n} = \sum_{i=1}^{n-1} h_{i,i+1} = \sum_{i=1}^{n-1} (2i - 1) = (n - 1)^2
\]