CS 4850: Wednesday, 4 March 2009

Random Walks: Probabilistic Interpretations of Voltage and Current

Scribes: Christie Brandt (ccb35) and Sara Tansey (sjt33)

1) **Review: Last lecture…**

a) The problem: modeling random walks on undirected graphs.

b) To converge to a stationary probability, an undirected graph must be:

i) Connected

(1) You can make an arbitrary graph connected by adding a "restart node" with an edge to every node in the graph; with a certain probability, you will take the edge to the restart node.

ii) Aperiodic (the greatest common denominator of the lengths of all cycles should be 1)

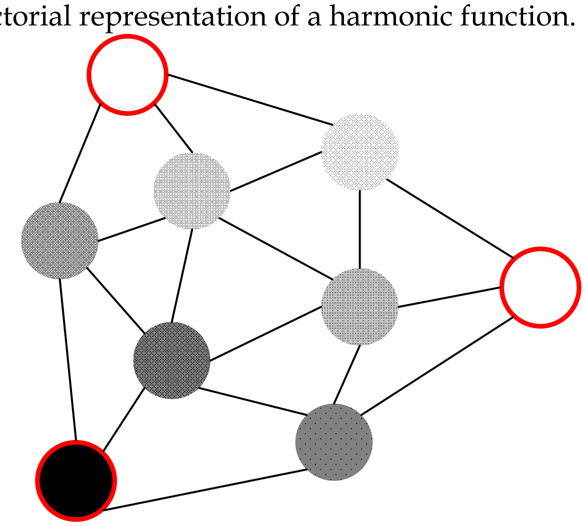
c) **Harmonic functions**: a harmonic function on an undirected graph designates exterior and interior points as follows: exterior points are given a fixed value. The value of an interior point is a weighted average of its neighbors.

i) A harmonic function takes on its maximum and minimum values on the boundary. (This makes sense, because any interior node is the average of the nodes around it)

ii) The difference of two harmonic functions is harmonic; in other words, if g and h are harmonic, then g-h is harmonic.

iii) Given a set of boundary conditions and a weight function, there is a unique harmonic function satisfying these boundary conditions.

iv) Pictorial representation of a harmonic function. Exterior nodes are circled in red.



d) **Analogy between Electrical Circuits and Random Walks:**

i) Phys E&M 101:

(1) Ohm's Law: v=ir (This is equivalent to i = vc, where c is the conductance)

(2) Kirchoff's Law: $\sum_y i_{xy} = 0$

ii) Each edge in the undirected graph is a resistor; an edge xy has resistance r_{xy}.

(1) The resistance r_{xy} = Pr(going to y | currently at x)

iii) Conductance (c_{xy}) = 1/ r_{xy}

(1) P_{xy} = Pr(going to y | currently at x) = $\frac{c_{xy}}{c_x} = \frac{c_{xy}}{\sum_y c_{xy}}$

2) **Changing order of conductance subscripts**—Claim: $c_xP_{xy} = c_yP_{yx}$

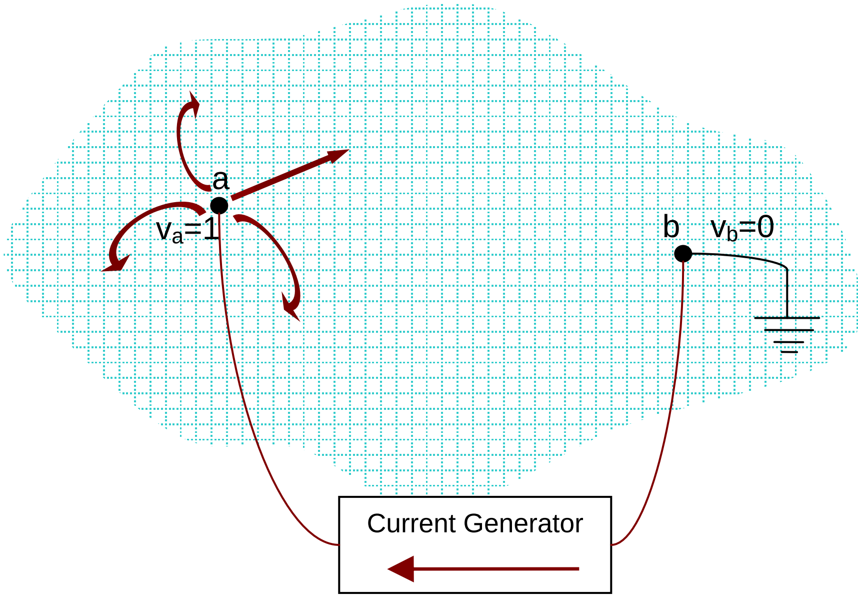
a) Proof: $c_x P_{xy} = c_x \frac{c_{xy}}{c_x} = c_{xy} = c_{yx} = c_y \frac{c_{yx}}{c_y} = c_y P_{yx}$

b) Therefore, $c_xP_{xy} = c_yP_{yx.}$ We will uses this identity in the proof of the interpretation of current.

3) **Interpretation of Voltage**:

a) Claim: v_x = Pr(a random walk, started at x, reaches a before b).

b) Setup: let us choose two vertices, a and b, from a random undirected graph. We will hook up a current generator to a, providing enough current so that $v_a = 1$. We also will ground b so that $v_b = 0$.



c) Voltage is a harmonic function. Proof:

i) Voltage v_x = voltage at vertex x

ii) $i_{xy} = \frac{v_x - v_y}{v_{xy}} = (v_x - v_y)c_{xy}$

iii) By Kirchoff's Law, $\sum_y i_{xy} = 0$.

iv) Substituting for i_{xy}, $\sum_y (v_x - v_y)c_{xy} = 0$

v) We can then rearrange this as: $\sum_y v_y c_{xy} = \sum_y v_x c_{xy} = v_x \sum_y c_{xy} = v_x c_x$

vi) Solving for v_x, we have: $v_x = \frac{\sum_y v_y c_{xy}}{c_x} = \sum_y \left(v_y \frac{c_{xy}}{c_x} \right) = \sum_y \left(v_y P_{xy} \right)$

vii) Notice that voltage is a harmonic function, since $v_x = \sum_y v_y P_{xy}$

d) P_x is a harmonic function. Proof:

i) P_x is the probability that a walk, started at x, reaches a before b.

ii) P_x therefore can be recursively defined by its neighbors: let y be the neighbors of x. Then: $P_x = \sum_y P_y P_{xy}$

e) Now, examine the functions for v_x and P_x. Also, recall that v_x and P_x have the same weights and boundary conditions; therefore, they must be the same.

f) Conclusion: if $v_b = 0$ and $v_a = 1$, then $v_x = P_x$ = Pr(a random walk, started at x, reaches a before b.)

4) **Interpretation of Current**:

Claim: when 1 A of current is injected into a, the amount of current that goes through an edge is equivalent to the net frequency with which a random walk from a to b goes through the edge.

a) edge xy.

b) Let u_x be the number of visits to vertex x on a walk from a to b before reaching b.

i) $u_b = 0$. (This makes sense—at b, we've already reached b…)

ii) For x ≠ a, b, $u_x = \sum_y u_y P_{yx}$

(1) (this is a recursive definition in which the set of y values are the neighbors of x.)

c) Rearranging this equation:

$$u_x = \sum_y u_y P_{yx}$$

$$\Rightarrow u_x = \sum_y u_y P_{xy} \frac{c_x}{c_y} \text{ (using our conductance identity to switch subscript order)}$$

$$\Rightarrow u_x = \sum_y u_y P_{xy} \frac{c_x}{c_y}$$

$$\Rightarrow \frac{u_x}{c_x} = \sum_y P_{xy} \frac{u_y}{c_y} \text{ (which is harmonic, with a boundary condition } \frac{u_b}{c_b} = 0\text{)}$$

d) Now let us set the voltage coming into a, v_a, as $v_a = \frac{u_a}{c_a}$. We are allowed to set the incoming voltage as anything we want; however, this choice means that since $\frac{u_x}{c_x}$ and v_x are both harmonic and have identical boundary conditions and weights, they are equivalent.

e) What is the current i_{xy}?

$$i_{xy} = (v_x - v_y)c_{xy} \text{ (Ohm's Law)}$$

$$= \left(\frac{u_x}{c_x} - \frac{u_y}{c_y}\right)c_{xy} \text{ (Because } v_x \text{ and } \frac{u_x}{c_x} \text{ represent the same harmonic function)}$$

$$= u_x \frac{c_{xy}}{c_x} - u_y \frac{c_{xy}}{c_y}$$

$$= u_x P_{xy} - u_y P_{yx}$$

i) What is the current into *a* from the current source?

(1) The expected number of times the walk leaves a is 1. Therefore, the current into *a* from the current source for $v_a = \frac{u_a}{c_a}$ is 1 A.

ii) Therefore, since $i_{xy} = u_x P_{xy} - u_y P_{yx}$, the current is the expected net number of times we leave the node x and go to y.

iii) (More detail on this proof next lecture)

Summary of Electrical Circuit Analogy:		
Electrical Property	***Identities***	***Probabilistic Interpretation***
Resistance (r_{xy})	• v = ir (Ohm's Law)	P_{xy} =Pr(going to y | currently at x)
Conductance (c_{xy})	• $c_{xy}=1/r_{xy}$ • $c_x P_{xy} = c_y P_{yx}$ • $P_{xy} = \frac{c_{xy}}{c_x} = \frac{c_{xy}}{\sum_y c_{xy}}$	1/ Pr(going to y | currently at x)
Voltage (v_x)	• $v_x = \sum_y v_y P_{xy}$	Pr(a random walk, started at x, reaches a before b)
Current (i_{xy})	• $\sum_y i_{xy} = 0$ (Kirchoff's Law) • $i_{xy} = (v_x - v_y)c_{xy}$ (Ohm's Law)	net frequency with which a random walk from a to b goes through edge x,y.

	• $i_{xy} = u_x P_{xy} - u_y P_{yx}$	