Monday 2 March - Lecture 19

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Random Walks on Graphs

If we walk for a very long time in a graph, what is the probability we are at a given vertex? We hope to find a stationary probability, in other words, we hope that the probability distribution converges.

Some Little Difficulties

Directed versus Undirected Graphs From a vertex, we pick edges uniformly at random to travel along. This presents some problems for directed graphs in the case of a vertex with only incoming edges, and a vertex with no incoming edges. We can circumvent this by requiring that the graph be strongly connected (i.e. there is a path from each vertex in the graph to every other vertex) or by adding a "restart" vertex, connected in both directions to all other vertices (sometimes, we'll also incorporate a bias against choosing an edge to the reset vertex). For undirected graphs, we only require that the graph is connected.

Figure 1: No outgoing edges / no incoming edges

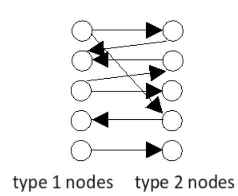

Figure 2: Problematic bipartite graph

Periodicity We say a graph is "**periodic**" if the gcd of cycle lengths is > 1. In a bipartite graph, for example, the probability of being at a given vertex won't converge, because depending on whether we are at an even or odd unit of time we'll be in one or the other sides. Thus, we also require that graphs be aperiodic (gcd of cycle length $= 1$), in both undirected and directed graphs.

Analogy with Electrical Networks

Terminology: edge (x, y) has resistance r_{xy} and conductance $C_{xy} = \frac{1}{r_{xy}}$. We assign a probability to each edge adjacent to a vertex, P_{xy}, the probability of traveling to vertex x from vertex y. By the analogy we've developed, this is $= \frac{C_{xy}}{C_x}$. C_x here is a normalization for vertex x, equal to the sum of conductances of its edges, $C_x = \sum_y C_{xy}$. We try to find solutions of $F = P^T F$, where P is the matrix of P_{ij} with rows normalized to sum to 1, and F is a matrix of vectors $(f_1, ..., f_n)$. Then f_x (the xth component of F) $= \frac{C_x}{C}$ (with C being the sum of all conductances, $\sum_i C_i$). If all $r_{xy} = 1$ (all edges present as 1-ohm resistors), then $C_x = d_x$ (the degree of the vertex), and $C = \sum_x d_x = 2m$ (m being the number of edges).

Above, we presume $f_x = \frac{C_x}{C}$, we'd like to justify this claim. Intuitively, to ask if we're at a given vertex x in the graph, we want to look all all vertices adjacent to x and the probability of jumping to x in the next step for an undirected graph with equal resistances. This is $f_x = \sum_y P_{yx} f_y = \sum_y \frac{C_{yx}}{C_y} \frac{C_y}{C} = \frac{1}{C} \sum_y C_{yx} = \frac{C_x}{C}$. For undirected graphs, this is relatively uninteresting. With d_x the degree of vertex x and m the number of edges in the graph, it is $= \frac{d_x}{2m}$).

Harmonic Functions

We have a graph, with certain vertices designated as boundary vertices. Define f such that if x is a boundary vertex, $f(x) =$ some boundary condition. For interior vertices, $f(x)$ is a weighted sum of values at adjacent vertices. This harmonic function will take on its minimum and maximum values at the boundary. If g and h are harmonic functions or solutions to harmonic functions, then $g - h$ also is. This implies a unique solution, because otherwise $g - h$ would be 0 on the boundary - thus $g = h$.

Developing the Analogy: 1 Let all edges be 1 ohm resistors, and pick two vertices in the graph, b and a. Connect a current generator to b and a and pump current such that the voltage at a, v_a, is 1, with v_b defined $= 0$. We claim that $\exists$ a vertex x, and v_x is the probability that a random walk starting at x reaches a before b.

Developing the Analogy: 2 Instead of pumping current to reach a desired voltage, use 1 amp of current. Then, the current at each edge is the frequency that the random walk travels through that edge.

We can now define the effective resistance the graph, and represent it with r_e. Replacing the graph by a single resistor r_e allows us to define an escape probability, the probability that random walk from a reaches b before returning to a. We want to show that the escape probability is a function of $r_e = \frac{C_e}{C_a} = \frac{1}{r_e C_a}$. Imagine we have a rectangular, infinite grid (so each vertex in the interior is connected to four others). We place b on the boundary of this graph. In the limiting case of the infinite grid in two dimensions, $r_e = \infty$, and the escape probability is 0. However, in 3 dimensions, there are so many infinite, parallel paths, that r_e is a finite number, and thus there is a probability of escape.