1 The Small-World Phenomenon

Any two individuals, who do not know each other in the United States, are likely to be connected through less than six acquaintances.

2 Experiment by Milgram

Task: A source person receives a letter, which is to be delivered to a target person in Massachusetts. If the source person does not know the target person directly, then he forwards the letter to a friend who is more likely to know the target person. Repeat until the target person receives the letter.

3 Jon Kleinberg’s Algorithm

The idea: Model the world as a 2-dimensional grid, in which each vertex is a person. There are two types of connections: local edges, in which a person knows his neighbors, represented by edges to adjacent vertices, and long distance edges, going from a vertex u to an arbitrary vertex v.

Start vertex: s
Current vertex: u
End vertex: t

Let’s define the probability of a long distance edge existing as:

$$Pr(u, v) \propto d^{-r}(u, v) = \frac{1}{d_{r}(u,v)}$$

where r is parameter range in $[0, \infty)$.

It takes $\log n$ to find a short path.

Case $r = 0$:
End point pick uniformly at random.

Case $r < 2$:
Surely there exists short path but no local algorithm.

Case $r = 2$:
There exists efficient algorithm to find the short path.

Case $r > 2$:
Maybe no short path exists.

Case $r = \infty$:
No long distance edges exist.

u is in phase j when distance $d(u,t)$ is in range $(2^j, 2^{j+1}]$. At each phase, finding short path takes $\log n$ time. Since we have $\log n$ phases, total expected time is $\log^2 n$. 
Lemma Select \( s \) and \( t \) uniformly at random on \( n \times n \) grid, \( s \) and \( t \) will be at least \( n/4 \) apart with prob \( \geq \frac{1}{2} \).

Proof Consider case \( r=0 \):
There are order of \( n \) nodes in the circle. There are order \( n^2 \) nodes total.
- \( \Pr(\text{long distance edge lands in circle}) = \frac{1}{n} \).
- \( \Pr(\text{not in circle}) = 1 - \frac{1}{n} = 1 - \frac{1}{n} \).
- \( \Pr(\text{none of } n^{1/2} \text{ long distance edges land in circle}) = (1 - \frac{1}{n})^{n^{1/2}} \).

As \( n \to \infty \),

\[
\lim_{n \to \infty} \left( 1 - \frac{1}{n} \right)^{n^{1/2}} = \left( 1 - \frac{1}{n} \right)^{n^{1/2} \frac{n^{1/2}}{n}} = \lim_{n \to \infty} e^{-\frac{1}{n^{1/2}}} = 1.
\]
Figure 3: At least with probability 1/2, s and t are at least n/4 apart

Figure 4: Number of nodes inside the circle.

4 Jon Kleinberg’s algorithm for r=2

But why does the r=2 algorithm work?

Lemma
For r=2, there exists constant c such that probability of long distance edges from u to v is at least \( c \frac{d^{-2}(u,v)}{\ln n} \).

Proof
\[
\Pr(\text{long distance edge from } u \text{ to } v) = \frac{\sum_{w \neq u} d^{-2}(u, w)}{d^{-2}(u, w)}
\]

Since upper bound on denominator implies lower on proportionality,
\[
\sum_{w \neq u} d^{-2}(u, w) \leq \sum_{i=1}^{2n-2} 4i \cdot \frac{1}{i^2} = 4 \sum_{i} \frac{1}{i} \leq c_1 \ln n
\]