CS4850

Lecture 16: February 23, 2009

Growth Model

At time t= 0, there are no vertices.
With each unit of time, add 1 vertex and with probability δ add an edge between two vertices selected uniformly at random.

Remember from previous lecture, degree distribution is given by:

$$P_k = \frac{(2\cdot\delta)^k}{(1+2\cdot\delta)^{k+1}}$$

<u>Generating function for component size</u>
Let $N_k(t)$ be the expected number of components of size k at time t when components are selected uniformly at random.

$$N_k(t) = a_k t$$

$$a_1 = \frac{1}{1+2\delta} \qquad a_k = \frac{\delta}{1+2k\delta}\sum_{j=1}^{k-1} j(k-j)a_j a_{k-j}$$

$$g(x) = \sum_1^{\infty} k a_k x^k$$

$$-x + \sum_1^{\infty} k a_k x^k + 2\delta x \sum_1^{\infty} k^2 a_k x^{k-1} = \delta \sum_1^{\infty} k x^k \sum_1^{k-1} j(k-j)a_j a_{k-j}$$
$$\Rightarrow -x + g(x) + 2\delta x g'(x) = 2\delta x g'(x) g(x)$$

Solving for g'(x) gives:

$$g'(x) = \frac{1}{2\delta}\frac{1-\frac{g(x)}{x}}{1-g(x)}$$

Now we want the phase transition.
We know that g'(1) is the average size of finite components.

$g(1) = \sum k a_k$ is the probability a vertex is in a finite component ($k a_k$ is the probability a vertex is in a component of size k)

If g(1) = 1, then there is no giant component.
If g(1) < 1, then (1-g(1)) is the faction of vertices in a giant component.
If δ=0, the average size of a component is 1.
$\delta_{critical}$: as a giant component appears, the average component size should decrease with a larger δ.
If $\delta > \delta_{critical}$, then g(1) < 1 and there exists a giant component

$g'(1) = \frac{1}{2\delta}$ is the average size of a component

If $\delta < \delta_{critical}$,

$$\lim_{x\to 1} g'(x) = \frac{1}{2\cdot\delta} \cdot \frac{\left[x\cdot(g'(x)) - (g(x))\right]}{x^2\cdot(g'(x))}\Big|_{x=1}$$

$$\Rightarrow \frac{1}{2\cdot\delta} \cdot \frac{g'(1) - g(1)}{g'(1)} = g'(1)$$

This solves to $g'(1) = \frac{1}{4\delta} \pm \frac{1}{4\delta}\sqrt{1-8\delta}$

We take the 'minus' solution.