Previously:

Let the generating function \( f(x) = \sum_{i=0}^{\infty} p_i x^i \), where \( p_i \) is the probability of having \( i \) children. Let \( m \) be the slope of \( f \) at 1 (i.e. \( m = f'(1) \)). As we showed in the previous lecture, you can think of \( m = 1 \) as being the threshold between infinite and non-infinite trees. We also showed in class that \( q = f(f(f(...))) \) is the probability of extinction.

Lemma: Let \( m = f'(1) \). If \( m \neq 1 \) then the expected size of an extinct family is finite. If \( m = 1 \) and \( p_1 = 1 \), then the tree is infinite and has regular degree, so there is no extinct family. If \( m = 1 \) and \( p_1 < 1 \), then the expected size is infinite.

Proof: Let \( z_i \) be the size of generation \( i \). The expected value of \( z_1 \) given extinction may differ from the expected value of \( z_1 \) in general. This makes sense, because if you know that a process is extinct, then the first generation is more likely to be small.

By Bayes rule, \( P(z_1 = k|\text{extinction}) = \frac{P(\text{extinction}|z_1=k)P(z_1=k)}{P(\text{extinction})} \). You can verify this statement by multiplying both sides by \( P(\text{extinction}) \) which yields the joint probability on both sides of the equation. Note that if \( z_1 = k \), the probability of extinction is the probability that each of the \( k \) children become extinct. Thus \( P(\text{extinction}|z_1=k) = q^k \) and \( P(z_1 = k|\text{extinction}) = p_kq^k \).

To verify that \( \sum_{k=0}^{\infty} P(z_1 = k|\text{extinction}) = 1 \), note that \( \sum_{k=0}^{\infty} p_k q^{k-1} = \frac{1}{q} \sum_{k=0}^{\infty} p_k q^k = \frac{f(q)}{q} \). But \( f(q) = q \), which proves the statement.

Now we calculate: \( E(z_1|\text{extinction}) = \sum_{k=0}^{\infty} kP(z_1 = k|\text{extinction}) = \sum_{k=0}^{\infty} kp_k q^{k-1} \). Note that \( f'(x) = \sum_{k=0}^{\infty} kp_k x^{k-1} \) and hence \( E(z_1|\text{extinction}) = f'(q) \).

What is the expected size of \( z_1 \)? \( \sum_{k=0}^{\infty} k p_k = \sum_{k=0}^{\infty} k p_k x^{k-1} |_{x=1} = f'(1) = m \).

Note that if \( m < 1 \), there are no infinite trees. Thus \( E(z_1|\text{extinction}) = E(z_1) = m \).
Expected size of generation \( i \) (slightly different than what was presented in lecture)

Intuitively, this value seems to be \( |E(z_1|\text{extinction})|^i = |f'(q)|^i \), but we need to be more rigorous.

Any member of generation \( i \) must have been a child from some vertex in generation \( i-1 \). So if we number the vertices from generation \( i-1 \), the number of vertices in generation \( i \) is the number of children from vertex 1, plus the number of children from vertex 2, etc. Furthermore, the number of children from a particular vertex in generation \( i-1 \) is distributed the same as \( z_1 \), since each vertex is an independent branching process. Putting these ideas together: since there are \( z_{i-1} \) vertices in generation \( i-1 \), we find \( z_i = \sum_{i=1}^{z_{i-1}} z_1 \). Note that \( z_{i-1} \) is a random variable, so we now need a technique that will allow us to calculate the expectation of this expression.

Let \( n \) be a fixed constant. Then by linearity of expectation, \( E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) \). If all of the \( X_i \) have the same distribution, then \( E(\sum_{i=1}^{n} X_i) = nE(X_1) \).

What do you do if the number of variables, \( n \) is a random variable? In this case, \( E(\sum_{i=1}^{n} X_i) = P(n = 1)E(X_1|n = 1) + P(n = 2)E(X_1 + X_2|n = 2) + P(n = 3)E(X_1 + X_2 + X_3|n = 3) + ... \). But the \( X_i \) are statistically independent, so applying linearity of expectation, \( E(\sum_{i=1}^{n} X_i) = P(n = 1)E(X_1) + 2P(n = 2)E(X_1 + X_2) + 3P(n = 3)E(X_1 + X_2 + X_3) + ... = E(X_1)[1P(n = 1) + 2P(n = 2) + 3P(n = 3) + ...] = E(n)E(X_1) \).

It follows that \( E(z_i|\text{extinction}) = E(\sum_{i=1}^{z_{i-1}} z_1|\text{extinction}) = E(z_{i-1}|\text{extinction})E(z_1|\text{extinction}) \). Therefore \( E(z_i|\text{extinction}) = [E(z_1)]^i = [f'(q)]^i \). Thus the expected size of a tree is \( E(\text{tree}) = \sum_{i=0}^{\infty} (f'(q))^i = \frac{1}{1-f'(q)} \).

If \( m < 1 \), then \( f(x) = x \) when \( x = 1 \). Therefore \( q = 1 \) and so \( f'(q) < 1 \). If \( m > 1 \), then \( f(x) = x \) at \( x = q \) and \( x = 1 \). Therefore the slope at \( q \) must have been less than 1. Thus if \( m \neq 1 \), the expected size of the tree is finite.

Generating functions

Here are some more uses for generating functions. Let \( a_0, a_1, a_2, ... \) denote the generating function \( g(x) = \sum_{i=0}^{\infty} a_i x^i \).

- \( a_1, 2a_2, 3a_3 = x \frac{d}{dx} (g(x)) = \sum_{i=0}^{\infty} ia_i x^{i-1} \).
- \( 1, 1, 1, 1 = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \).
- \( 0, 1, 2, 3 = \sum_{i=0}^{\infty} i x^{i-1} = \frac{d}{dx} \left( \sum_{i=0}^{\infty} x^i \right) = \frac{x}{(1-x)^2} \).

Choosing items Suppose \( A \) can be selected 0 or 1 times: generating function \( 1 + x \). \( B \): 0, 1, or 2 times; \( 1 + x + x^2 \). \( C \): 0, 1, 2, 3 times; \( 1 + x + x^2 + x^3 \). How many ways can you select five items? Multiplying the three generating functions gives \( (1+x)(1+x+x^2)(1+x+x^2+x^3) = 1 + 5x + 5x^2 + 6x^3 + 5x^4 + 3x^5 + x^6 \). Thus there are 3 ways to choose five items, the coefficient of \( x^5 \). These are CCCBB, CCCBA, and CCBB.

Making change: How many ways are there to make 23 cents with pennies, nickels, and dimes?

- Pennies: \( 1 + x + x^2 + ... \)
- Nickels: \( 1 + x^5 + x^{10} + ... \)
- Dimes: \( 1 + x^{10} + x^{20} + ... \)

Multiplying these three generating functions together, we find that the coefficient of \( x^{23} \) is 9. Indeed, there are nine ways: \( ddppp, dnnppp, dnp^3, dp^{13}, n^4p^3, n^3p^8, n^2p^{13}, np^{18}, \) and \( p^{23} \).