CS 4850 – Lecture 13

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Scribes: Justin Choi and Sean Sullivan

<u>Branching Process</u> (review from 2/13/09 lecture)

Given P_0, P_1, P_2, … where P_n = probability of n children

Generating function $f(x) = \sum^{\infty}_{i=0} P_i * x^i$

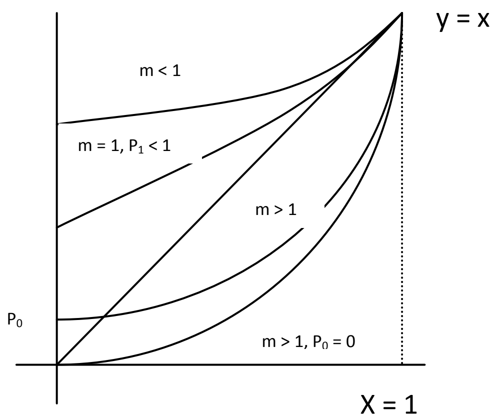
$f_1(x) = f(x)$

$f_{j+1}(x) = f_j(f(x))$

x_1 $\quad f(x) = a_0 + a_1x + a_2x^2 + \ldots$

$x_1 + x_2$ $\quad f^2(x) = a_0a_0 + (a_0a_1 + a_1a_0)x + (a_0a_2 + a_1a_1 + a_2a_0)x^2 + \ldots$

$f_j(x) = b_0 + b_1x + b_2x^2 + \ldots = b_0 + b_1f(x) + b_2f^2(x) + \ldots$



q is the solution of $f(x) = x$

Slope of f(x) at x = 1 (m), 3 cases:

$m < 1$ * then $q = 1$

$m = 1$ → a) $P_1 = 1$ b) $P_1 < 1$

$m > 1$ → a) $P_0 > 0$ b) $P_0 = 0$ * only in this case does $q < 1$ converge

$\lim_{j \to \infty} f_j(x) = c_0 + c_1x + c_2x^2 + \ldots$

if limit is constant, probability of $i > 0$ children is 0 (because $c_1x + c_2x^2 + \ldots \to 0$) and probability of 0 children is c_0.

Branching Processes: Expected Size of Finite Component

All extinct trees have finite size.
Let S_i = Pr(tree size = i) (note: this implies $\sum_{i=0}^{\infty} S_i = 1$)

E = Expected tree size = $\sum_{i=0}^{\infty} iS_i$
Despite all extinct trees having finite size, the expected size of the trees may not be finite
Example:
$S_i = \frac{6}{\pi^2}\left(\frac{1}{i^2}\right)$
Note : $\sum_{i=0}^{\infty} S_i = 1$, a famous result due to Euler in 1735, this was written incorrectly as $\frac{\pi}{6}$ in class
See: http://en.wikipedia.org/wiki/Basel_problem
E $=\frac{6}{\pi^2}\sum_{i=0}^{\infty}\frac{1}{i}$, which diverges.

What is the probability of extinction if the 1st generation has k children?
q= probability of extinction (determined based on the P_i = probability of i children, see previous lecture)

Each of the k children has probability q of becoming extinct. The entire tree will become extinct if and only if all k children become extinct. These events are independent of each other, so the probability that the entire tree becomes extinct is q^k.

Note: $\sum_{k=0}^{\infty} \mathrm{P_k q^k} = q$

Expected Size of 1st generation over all extinct trees
Z_i = Random variable denoting size of ith generation

$$\Pr(Z_i = k|extinction) = \frac{\Pr(Z_i = k) \times \Pr(extinction|Z_i = k)}{\Pr(extinction)}$$

$$\Pr(Z_1 = k|extinction) = \frac{\mathrm{P_k q^k}}{q} = \mathrm{P_k q^{k-1}}$$