Branching Process (review from 2/13/09 lecture)

Given \( P_0, P_1, P_2, \ldots \) where \( P_n \) = probability of \( n \) children

Generating function \( f(x) = \sum_{i=0}^{\infty} P_i x^i \)

\[ f_1(x) = f(x) \]

\[ f_{i+1}(x) = f_i(f(x)) \]

\[ x_1 \quad f(x) = a_0 + a_1 x + a_2 x^2 + \ldots \]

\[ x_1 + x_2 \quad f^2(x) = a_0 a_0 + (a_0 a_1 + a_1 a_0) x + (a_0 a_2 + a_1 a_1 + a_2 a_0) x^2 + \ldots \]

\[ f_i(x) = b_0 + b_1 x + b_2 x^2 + \ldots = b_0 + b_1 f(x) + b_2 f^2(x) + \ldots \]

Slope of \( f(x) \) at \( x = 1 \) (m), 3 cases:

\( m < 1 \) * then \( q = 1 \)

\( m = 1 \to a) \ P_1 = 1 \ b) \ P_1 < 1 \)

\( m > 1 \to a) \ P_0 > 0 \ b) \ P_0 = 0 \quad * \text{only in this case does } q < 1 \text{ converge} \)

\[ \lim_{n \to \infty} f_i(x) = c_0 + c_1 x + c_2 x^2 + \ldots \]

if limit is constant, probability of \( i > 0 \) children is 0 (because \( c_1 x + c_2 x^2 + \ldots \to 0 \)) and probability of 0 children is \( c_0 \).
Branching Processes: Expected Size of Finite Component

All extinct trees have finite size. Let \( S_i = \Pr(\text{tree size} = i) \) (note: this implies \( \sum_{i=0}^{\infty} S_i = 1 \))

\[
E = \text{Expected tree size} = \sum_{i=0}^{\infty} iS_i
\]

Despite all extinct trees having finite size, the expected size of the trees may not be finite

Example:

\[
S_i = \frac{6}{\pi^2} \left( \frac{1}{i^2} \right)
\]

Note: \( \sum_{i=0}^{\infty} S_i = 1 \), a famous result due to Euler in 1735, this was written incorrectly as \( \frac{\pi}{6} \) in class


\[
E = \frac{6}{\pi^2} \sum_{i=0}^{\infty} \frac{1}{i^2}, \text{ which diverges.}
\]

What is the probability of extinction if the 1st generation has \( k \) children?

\( q = \) probability of extinction (determined based on the \( \Pr_i = \) probability of \( i \) children, see previous lecture)

Each of the \( k \) children has probability \( q \) of becoming extinct. The entire tree will become extinct if and only if all \( k \) children become extinct. These events are independent of each other, so the probability that the entire tree becomes extinct is \( q^k \).

Note: \( \sum_{k=0}^{\infty} P_k q^k = q \)

Expected Size of 1st generation over all extinct trees

\( Z_i = \) Random variable denoting size of \( i \)th generation

\[
\Pr(Z_i = k|\text{extinction}) = \frac{\Pr(Z_i = k) \times \Pr(\text{extinction}|Z_i = k)}{\Pr(\text{extinction})}
\]

\[
\Pr(Z_1 = k|\text{extinction}) = \frac{P_k q^k}{q} = P_k q^{k-1}
\]