CS 4850 LECTURE 21 - MARCH 6, 2009

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RANDOM WALKS AND PROBABILISTIC INTERPRETATION OF CURRENT

Proof of Interpretation of Current
Define u_x as the number of visits to vertex x on a random walk from vertex a to vertex b before reaching b. Then,

$$u_b = 0, \forall x \neq a, b : u_x = \sum_y u_y P_{yx}$$

$$P_{yx} = \frac{C_{yx}}{C_y} = \frac{C_{xy}}{C_y} = \frac{C_x}{C_y}\frac{C_{xy}}{C_x} = \frac{C_x}{C_y}P_{xy} \Rightarrow C_x P_{xy} = C_y P_{yx}$$

$$u_x = \sum_y u_y P_{yx} = \sum_y u_y \frac{C_x P_{xy}}{C_y} \Rightarrow \frac{u_x}{C_x} = \sum_y \frac{u_y}{C_y} P_{xy}$$

Thus, $\frac{u_x}{C_x}$ satisfies the same hamonic equation as v_x.
Let's check the boundary conditions: $\frac{u_b}{C_b} = 0 = v_b$, so set $v_a = \frac{u_a}{C_a}$

$$i_{xy} = (v_x - v_y)C_{xy} = \left(\frac{u_x}{C_x} - \frac{u_x}{C_x}\right)C_{xy} = \frac{C_{xy}}{C_x}u_x - \frac{C_{yx}}{C_y}u_y = P_{xy}u_x - P_{yx}u_y$$
$$= \text{net traversals of edge } xy \text{ in a walk from } a \text{ to } b \text{ before reaching } b$$

Sum of net traversals of each edge at $a = 1$, and since the sum of traversals equals the sum of currents at a as shown above, therefore the sum of currents at a is 1.

ESCAPE PROBABILITY

Call $v_b = 0, v_a = IR_{eff} \Rightarrow R_{eff} \triangleq \frac{v_a}{I}, C_{eff} = \frac{1}{R_{eff}}$
Escape Probability is the probability that a walk starting at a reaches b before returning to a.
Need to show that $P_{esc} = \frac{C_{eff}}{C_a}$

$$i_a = \sum_y (v_a - v_y)C_{xy} = \sum_y (1 - v_y)\frac{C_{ay}}{C_a}C_a \text{ (when } v_a = 1)$$
$$= C_a \sum_y \left(\frac{C_{ay}}{C_a} - v_y \frac{C_{ay}}{C_a}\right) = C_a(1 - \sum_y v_y P_{ay})$$

$\sum_y v_y P_{ay}$ = Probability of walk starting at a coming back to a before reaching b,
so $i_a = C_a(1 - \text{Prob returning}) = C_a P_{esc}$

$$\therefore i_a = C_a P_{esc}$$

$$i_a = v_a C_{eff} = C_{eff} \Rightarrow C_{eff} = C_a P_{esc} \Rightarrow \frac{C_{eff}}{C_a} = P_{esc}$$

1-D case: $R_{eff} = \infty \Rightarrow C_{eff} = P_{esc} = 0$
2-D case:
Lower bound on resistance from a to b
Short out resistors in boxes around a
Number of resistors $= 4 * \sum$(Odd numbers)

$$R_{eff} = \frac{1}{4}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) = \Theta(\ln n) \rightarrow \infty$$

3-D case:
To Be Continued ...