Increasing Properties

Q is an increasing property of a graph G if when a graph has property Q, any graph obtained from G by adding edges has the property Q.

Notation

Let $p(n, \varepsilon)$ be the function $p(n)$ such that the probability $N_{p(n)}$ has property Q equal $\varepsilon$. $N_p = N(n, p)$

For any $\varepsilon > 0$, there exists a constant $m$ such that:

$P(n, 1 - \varepsilon) \leq mP(n, \varepsilon)$

Theorem

Every increasing property of $N_p$ has a threshold.

Proof

Let $0 < \varepsilon < \frac{1}{2}$ and let $m$ be an integer such that $(1 - \varepsilon)^m \leq \varepsilon$.

Consider union of $m$ independent copies of $N_{p(n, \varepsilon)}$. The union is equivalent to a single $N_q$ where we select an integer with a higher probability.

$q = 1 - [1 - P(n, \varepsilon)]^m$
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For example:

$$P(n, \epsilon) = \frac{3}{n}; N_p = \{5, 17, 81\}; N_q = \{10, 11\}; N_q \subset \{5, 10, 11, 17, 81, \ldots\}$$

$$q = 1 - [1 - P(n, \epsilon)]^m \leq m P(n, \epsilon)$$

By the previous inequality and the fact that Q is an increasing property, we have:

A. \(\text{Prob}(N_q \in Q) \leq \text{Prob}(N_{mp(\epsilon)} \in Q)\)

Since Q is an increasing property if one or more \(N_p(\epsilon)\) has property then \(N_q\) has property.

If \(N_q\) does not have property then for all \(N_p(\epsilon), N_q(\epsilon)\) does not have property.

$$\forall N_{p(\epsilon)} : N_{p(\epsilon)} \notin Q$$

$$\text{Prob}(N_q \notin Q) \leq \text{Prob}(\forall N_{p(\epsilon)} : N_{p(\epsilon)} \notin Q)$$

It is possible none of \(N_{p(\epsilon)}\) have Q, but \(N_q \in Q\).

Example if the property is that the set has 2 elements, then all \(N_p(\epsilon)\) have one element, but \(N_q\) might have more.

$$\text{Prob}(N_q \notin Q) \leq [1 - \text{Prob}(N_{p(\epsilon)} \in Q)]^m \leq (1 - \epsilon)^m \leq \epsilon$$

B. \(\text{Prob}(N_q \in Q) \geq 1 - \epsilon\)

Consequently we can sandwich the \(\text{Prob}(N_q \in Q)\) and conclude:

$$1 - \epsilon \leq \text{Prob}(N_q \in Q) \leq \text{Prob}(N_{mp(\epsilon)} \in Q)$$

C. Therefore, \(m(p(\epsilon)) \geq p(1 - \epsilon)\)

For \(0 < \epsilon < .5\): \(P(\epsilon) \leq P(0.5) \leq P(1 - \epsilon) \leq mp(\epsilon)\)

Since \(P(1/2)\) is between \(P(\epsilon)\) and some constant \(m\) times \(P(\epsilon)\), \(P(1/2)\) is asymptotically equivalent of \(P(\epsilon)\).

\(P(1/2)\) is a threshold.

\([P(\epsilon), P(1 - \epsilon)]\) is bounded by a constant.
Emergence of Cycles

Let \( x \) be the number of cycles in a graph, \( G(n,p) \). Define Cycles to length 3 or larger, removing self-loops and two nodes pointing to each other.

\[
x = \sum_{k=3}^{n} \binom{n}{k} \frac{(k-1)!}{2}
\]

\[
E(x) = \sum_{k=3}^{n} \binom{n}{k} \frac{(k-1)!}{2} p^k
\]

\[
E(X) \leq \sum_{k=3}^{n} \frac{(np)^k}{2k}
\]

If \( p << \frac{1}{n} \) \( E(x) \to 0 \), we can surely a graph selected at random, almost surely has no cycle.

\( p = \frac{c}{n} \quad c > 1 \quad E(x) \) goes to \( \infty \)

\( c < 1 \quad E(x) \) converges

Second moment method:

\( p << \frac{1}{n} \) no cycles

\( p = \frac{c}{n} \quad c < 1 \quad \) finite number of cycles

\( p = \frac{c}{n} \quad c > 1 \quad \) infinite number of cycles