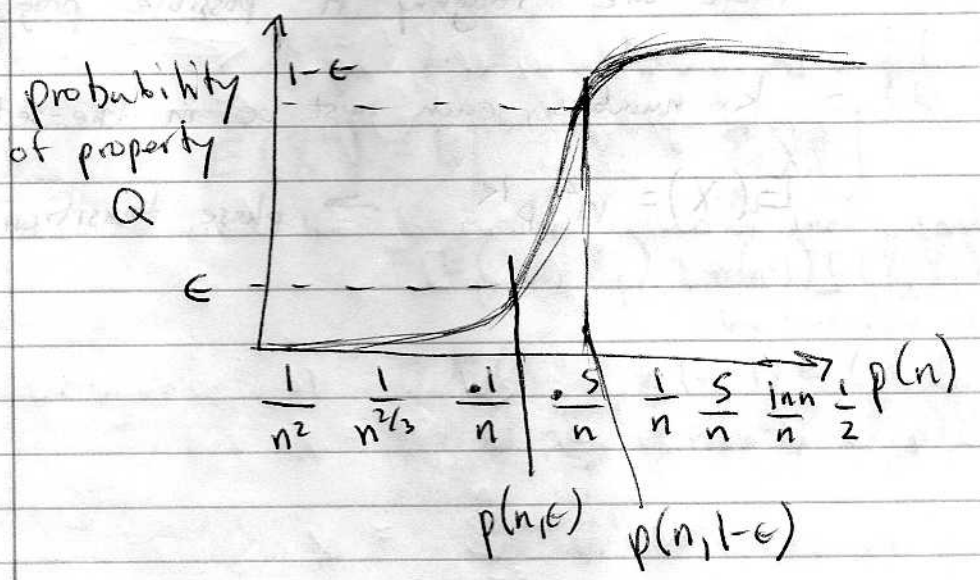
CS 485 (scribe!) 2-5-07

Increasing properties

<u>defn</u>: Q is an increasing property of a graph G if, when a graph has Q, any graph obtained from G ~~has~~ by adding edges also has Q

let $p(n, \epsilon)$ be the function $p(n)$ s.t. the probability $N_{p(n)}$ has property Q equals ϵ



we will prove $\forall \epsilon > 0 \ \exists$ a constant m s.t.

$$p(n, 1-\epsilon) \le m\, p(n, \epsilon)$$

i.e., transition from not having the property to having the probability is within a constant factor

<u>thm</u> Every increasing property of N_p has a threshold

Proof: Let $0 < \epsilon < \frac{1}{2}$
and $m \in$ integer such that $(1-\epsilon)^m \leq \epsilon$

consider union of m independent copies of ~~$N_{p(\epsilon)}$~~ $N_{p(n,\epsilon)}$

The union is equivalent to a single N_q where we select an integer with a higher probability $q = 1 - [1 - p(n,\epsilon)]^m$

$$q \leq m\, p(n,\epsilon)$$ from 1st term of binomial expansion

by the above inequality and the fact that Q is an increasing property

(A) $$\text{Prob}(N_q \in Q) \leq \text{Prob}(N_{mp(\epsilon)} \in Q)$$

Since Q is an increasing property
if 1 or more of the $N_{p(\epsilon)}$ has Q
then N_q has Q

taking contrapositive
$$N_q \notin Q \Rightarrow \forall N_{p(\epsilon)}.\ N_{p(\epsilon)} \notin Q$$

$$\Pr(N_q \notin Q) \leq \Pr(\forall N_{p(\epsilon)}.\ N_{p(\epsilon)} \notin Q)$$
$$\leq \Big[1 - \underbrace{\text{Prob}(N_{p(\epsilon)} \in Q)}_{\text{by definition, equals } \epsilon}\Big]^m$$
$$\leq (1-\epsilon)^m \leq \epsilon$$

(B) $\text{Prob}(N_q \in Q) \geq 1 - \epsilon$

$$\text{Prob}(N_{mp(\epsilon)} \in Q) \underset{(A)}{\geq} \text{Prob}(N_q \in Q) \underset{(B)}{\geq} 1 - \epsilon$$

$$\therefore \quad mp(\epsilon) \geq p(1-\epsilon) \quad (C)$$

for $0 < \epsilon < \frac{1}{2}$

$$p(\epsilon) \leq p(\tfrac{1}{2}) \leq p(1-\epsilon) \overset{(C)}{\leq} mp(\epsilon)$$

$\longrightarrow$ $p(\frac{1}{2})$ is asymptotically equivalent to $p(\epsilon)$

$\longrightarrow$ $p(\frac{1}{2})$ is a threshold

and the transition occurs on the interval $[p(\epsilon), p(1-\epsilon)]$

Emergence of cycles

let X = # of cycles in a graph $G(n,p)$

$$X = \sum_{k=3}^{n} \#\text{ of cycles of length } k$$

$$= \sum_{k=3}^{n} \underbrace{\binom{n}{k}}_{\text{choose vertices}} \underbrace{(k-1)!}_{\text{choose order}} \cdot \underbrace{\frac{1}{2}}_{\text{direction doesn't matter}}$$

$$E(X) = \sum_{k=3}^{n} \binom{n}{k} \frac{(k-1)!}{2} p^k$$

$$\leq \sum_{k=3}^{n} \frac{(np)^k}{2k}$$

$p \ll \frac{1}{n} \Rightarrow E(X) \to 0 \Rightarrow$ almost surely, a random graph has no cycle

if p is asymptotically equal to $\frac{c}{n}$ (w/ c = constant)

$$c > 1 \Longrightarrow E(x) \to \infty$$
$$c < 1 \Longrightarrow E(x) \text{ converges}$$

a second moment method will show

$p << \frac{1}{n}$ no cycles

$p \approx \frac{c}{n}$ $c < 1$ finite # of cycles

$p \approx \frac{c}{n}$ $c > 1$ infinite # of cycles