Increasing properties

defn: $Q$ is an increasing property of a graph $G$ if, when a graph has $Q$, any graph obtained from $G$ by adding edges also has $Q$.

Let $p(n, \varepsilon)$ be the function $p(n)$ s.t. the probability $Np(n)$ has property $Q$ equals $\varepsilon$.

\[
\begin{align*}
\text{probability} & \quad 1 - \varepsilon \\
\text{of property} & \quad Q \\
\varepsilon & \quad \frac{1}{n^2} \quad \frac{1}{n^{3/2}} \quad \frac{1}{n} \quad \frac{1}{n^{1/2}} \quad \frac{1}{n} \quad \frac{1}{n^{1/2}} \quad \frac{1}{n^2} \\
\Rightarrow p(n, \varepsilon) & \quad \Rightarrow p(n, 1 - \varepsilon)
\end{align*}
\]

We will prove $\forall \varepsilon > 0 \exists$ a constant $m$ s.t.

\[p(n, 1 - \varepsilon) \leq m p(n, \varepsilon)\]

i.e., transition from not having the property to having the probability is within a constant factor.
Thm: Every increasing property of $N_p$ has a threshold

Proof: Let $0 < \varepsilon < \frac{1}{2}$ and $m \in \mathbb{N}$ integer such that $(1 - \varepsilon)^m \leq \varepsilon$

Consider union of $m$ independent copies of $N_p(n, \varepsilon)$

The union is equivalent to a single $N_q$ where we select an integer with a higher probability $q = 1 - \left(1 - p(n, \varepsilon)\right)^m$

$q \leq mp(n, \varepsilon)$ from 1st term of binomial expansion

by the above inequality and the fact that $Q$ is an increasing property

$\mathbb{P}(N_q \notin Q) \leq \mathbb{P}(N_{mp(e)} \notin Q)$

since $Q$ is an increasing property if 1 or more of the $N_p(e)$ has $Q$ then $N_q$ has $Q$

taking contrapositive

$N_q \notin Q \Rightarrow \forall N_p(e), N_p(e) \notin Q$

$\mathbb{P}(N_q \notin Q) \leq \mathbb{P}(\forall N_p(e), N_p(e) \notin Q)$

$\leq \left[1 - \mathbb{P}(N_p(e) \notin Q)\right]^m$

by definition, equals $\varepsilon$

$\leq (1 - \varepsilon)^m \leq \varepsilon$
\[ \text{Prob}(N_q < \theta) \geq 1 - \epsilon \]

\[ \text{Prob}(N_{m'p}(\theta) < \theta) \geq \text{Prob}(N_q < \theta) \geq 1 - \epsilon \]

\[ \therefore \quad m_p(\epsilon) \approx p(1 - \epsilon) \]

for \( 0 < \epsilon < \frac{1}{2} \)

\[ p(\epsilon) = p\left(\frac{1}{2}\right) \approx p(1 - \epsilon) \approx m_p(\epsilon) \]

\[ \Rightarrow \quad p\left(\frac{1}{2}\right) \text{ is asymptotically equivalent to } p(\epsilon) \]

\[ \Rightarrow \quad p\left(\frac{1}{2}\right) \text{ is a threshold} \]

and the transition occurs on the interval \([p(\epsilon), p(1 - \epsilon)]\)

**Emergence of cycles**

let \(X = \# \text{ of cycles in a graph } G(n, p)\)

\[ X = \sum_{k=3}^{n} \# \text{ of cycles of length } k \]

\[ = \sum_{k=3}^{n} \binom{n}{k} (k-1)! \cdot \frac{1}{2^k} \]

choose, choose direction, vertices order doesn't matter

\[ E(K) = \sum_{k=3}^{\infty} \frac{\binom{n}{k}(k-1)!}{2^k} p^k \]

\[ \leq \sum_{k=3}^{\infty} \frac{n^k (np)^k}{2k} \]

\[ p < \frac{1}{n} \Rightarrow E(X) \to 0 \Rightarrow \text{almost surely, a random graph has no cycle.} \]
if \( p \) is asymptotically equal to \( \frac{c}{n} \) (w/ \( c = \text{const} \))

\[
\begin{align*}
C > 1 & \implies E(x) \to \infty \\
C < 1 & \implies E(x) \text{ converges}
\end{align*}
\]

a second moment method will show

\[
\begin{align*}
p &< \frac{1}{n} \quad \text{no cycles} \\
p &\approx \frac{c}{n} \quad C < 1 \quad \text{finite \# of cycles} \\
p &\approx \frac{c}{n} \quad C > 1 \quad \text{infinite \# of cycles}
\end{align*}
\]