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$$I(d) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-(x_1+x_2^2+x_3^2+\cdots)}\, dx_d \cdots dx_2\, dx_1 = \pi^{d/2}$$

$$= \int_{S_d}\int_0^{\infty} e^{-r^2} r^{d-1}\, dr\, d\Omega = A(d)\,\frac{1}{2}\,\Gamma\left(\frac{d}{2}\right)$$

$$A(d) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$V(d) = \int_{S_d}\int_0^1 r^{d-1}\, d\Omega\, dr = A(d)\cdot\frac{1}{d} = \frac{2\pi^{d/2}}{d\,\Gamma\left(\frac{d}{2}\right)}$$

If $d=2$

$$V(2) = \pi$$

If $d=3$

$$V(3) = \frac{2}{3}\cdot\frac{\pi^{3/2}}{\Gamma(3/2)} \qquad \boxed{\Gamma(3/2) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}}$$

$$= \frac{2}{3}\cdot\frac{\pi^{3/2}}{\sqrt{\pi}/2} = \frac{4}{3}\pi$$

How does the volume grow?

$$V(d) = \frac{2\pi^{d/2}}{d\,\Gamma(d/2)} \approx \frac{(\text{Const})^d}{d!}$$

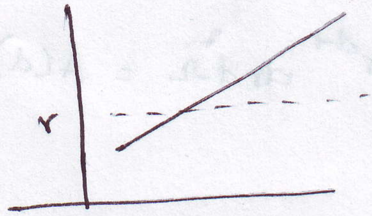
as $d \to \infty$

$$\frac{(\text{Const})^d}{d!} \to 0$$

How to find the max?

* MATLAB (≈ 5)

$\frac{v(d)}{v(d+1)}$ --- grows like this →



if $r \neq 1$ then eq. becomes

$$v(d) = \frac{2\pi^{d/2} \cdot r^d}{d\,\Gamma(d/2)} \quad \text{— ①}$$

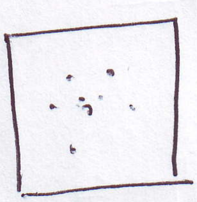
Stirling approx $= \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$

So ① becomes
$$\frac{(\text{const})^d}{\left(\frac{d}{2e}\right)^{d/2} \sqrt{2\pi \frac{d}{2}}} \cdot r^d \simeq \frac{(\text{const})^d}{d^{d/2}} \cdot r^d \neq 1 \quad \text{— ②}$$

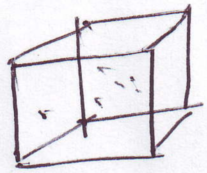
if radius $= \sqrt{d}$ then

② becomes
$$\frac{(\text{const})^d}{d^{d/2}} \cdot \left(\frac{\sqrt{d}}{\text{const}}\right)^d \simeq 1$$

In 2 Dimensions



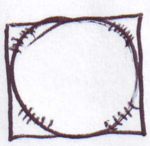
difference in ~~distance between~~ ~~smal~~ closest distance and farthest distance is quite alot but when $d=1000$



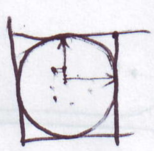
$$dist(z_1, z_2) = \sqrt{\sum_{i=1}^{d} (z_{1i} - z_{2i})^2}$$

generate pt in high dimensions, the distance between all points is constant.

To make them uniformly distributed in space, generate nos in sphere.



(i)



(but vol of sphere goes to 0)

(ii)

case i) If we project all the points from cube to sphere, then they would be concentrated in a few areas

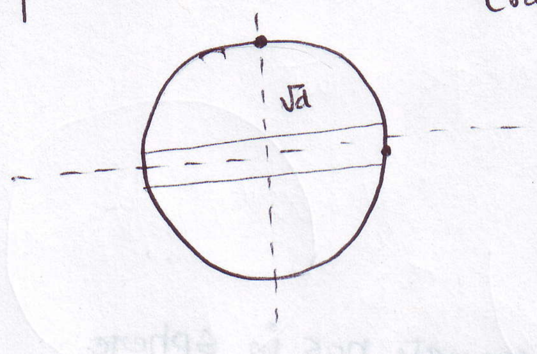
case ii) If we first distribute the points on the sphere and then project it back to cube. But at higher dimensions volume of sphere goes to 0.

generate $x_i \leftrightarrow \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2}$

$$[x_1, x_2 \dots x_d] = \frac{1}{\sqrt{2\pi d}} \cdot e^{-\frac{x_1^2 + x_2^2 + \dots x_d^2}{2}}$$

[Now uniformly distributed in space]

How far apart are random points?
(radius $\rightarrow \sqrt{d}$)

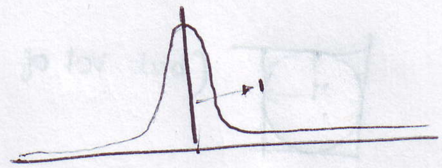


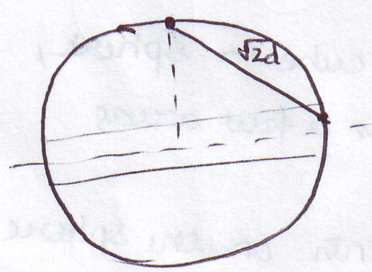
We need to select 2 points on the sphere. One on the North pole, the other in a band near the equator.
This is due to the fact that most points are concentrated near the equator ~~in a~~

$\Downarrow$

$\Rightarrow$ project the points on

distribution is $\frac{1}{2\pi} e^{-x^2/2} \longrightarrow$





distance between them is $\sqrt{2d}$.

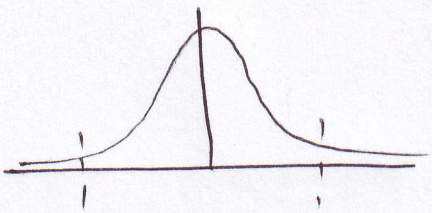
Two seperate process.

⇒ Two gaussian distribution

⇒ noisy generators.

How do we know which gaussian generated which points?

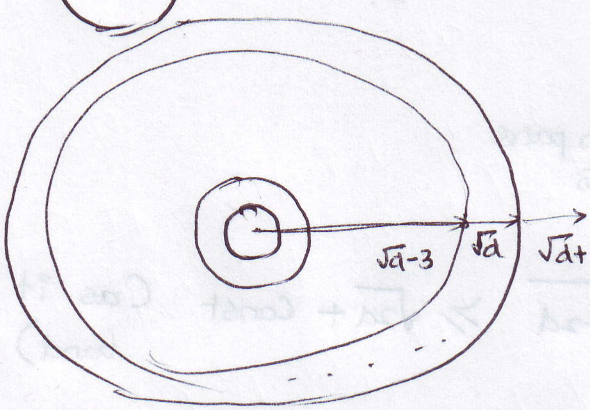
One Gaussian One D



at high dimension.

small sphere.

probability mass $\to 0$ [as volume $\to 0$ at high D]



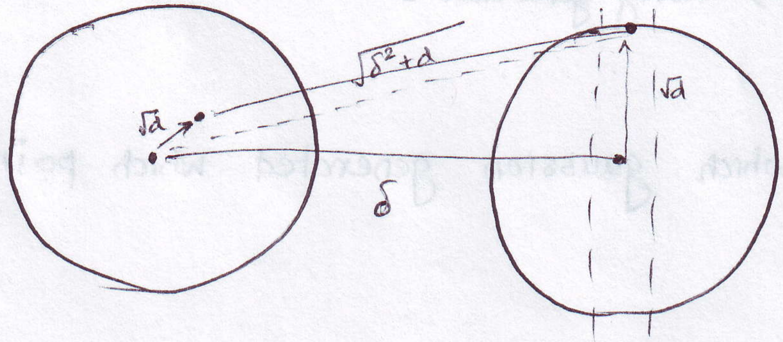
⇒ No points within $\sqrt{d}-3$. (as $V \to 0$)

⇒ as we near $\sqrt{d}$ ~~&~~ get some points.

⇒ as radius $> \sqrt{d}+3$. No points. [as probability dist $\to 0$]

Two points / same gaussian . they are $\sqrt{2d}$ apart

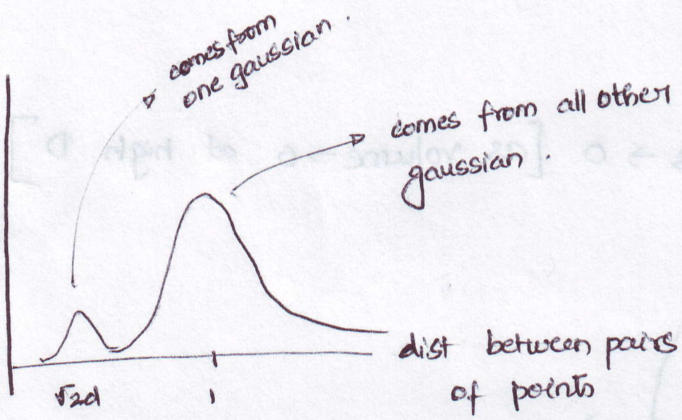
Two or different gaussian .



[band is (d-2) dimensional]

distance between 2 points .

$$= \sqrt{d+(\delta^2+d)} = \sqrt{\delta^2+2d}$$



$\Downarrow$

This works only when $\sqrt{\delta^2+2d} \geq \sqrt{2d} + \text{Const}$. (as it is in the band) .

$$\sqrt{\delta^2+2d} = \sqrt{2d}\left(1+\frac{\delta^2}{2d}\right)^{1/2}$$

$$= \sqrt{2d}\left[\cancel{1} + \frac{1}{2}\frac{\delta^2}{2\delta} - \right] \geq \cancel{\sqrt{2d}} + \text{Const}$$

$$\frac{\delta^2}{2}\sqrt{2d} \geq \text{Const}$$

$$\delta^2 \geq \text{Const} \cdot 2\sqrt{2d}$$

$$\delta = \text{const}\ d^{1/4}$$

If $d = 10{,}000$

then $\delta = 10$

Simple algorithm produces this, but ~~some~~ other algorithms do better.